Passage of Lévy Processes across Power Law Boundaries at Small Times*

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Abstract

We wish to characterise when a Lévy process X_t crosses boundaries like t^κ, $\kappa > 0$, in a one or two-sided sense, for small times t; thus, we enquire when $\limsup_{t\downarrow 0} |X_t|/t^\kappa$, $\limsup_{t\downarrow 0} X_t/t^\kappa$ and/or $\liminf_{t\downarrow 0} X_t/t^\kappa$ are almost surely (a.s.) finite or infinite. Necessary and sufficient conditions are given for these possibilities for all values of $\kappa > 0$. Often (for many values of κ), when the limsups are finite a.s., they are in fact zero, as we show, but the limsups may in some circumstances take finite, nonzero, values, a.s. In general, the process crosses one or two-sided boundaries in quite different ways, but surprisingly this is not so for the case $\kappa = 1/2$. An integral test is given to distinguish the possibilities in that case. Some results relating to other norming sequences for X, and when X is centered at a nonstochastic function, are also given.

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1 Introduction

Let $X = (X_t, t \geq 0)$ be a Lévy process with characteristic triplet (γ, σ, Π), where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and the Lévy measure Π has $\int (x^2 \wedge 1)\Pi(\mathrm{d}x)$ finite. See [1] and [15] for basic definitions and properties. Since we will only be concerned with local behaviour of X_t, as $t \downarrow 0$, we can ignore the "big jumps" in X (those with modulus exceeding 1, say), and write its characteristic exponent, $\Psi(\theta) = \frac{1}{t} \log E \mathrm{e}^{\mathrm{i}\theta X_t}$, $\theta \in \mathbb{R}$, as

$$\Psi(\theta) = \mathrm{i}\gamma\theta - \sigma^2\theta^2/2 + \int_{[-1,1]} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x)\Pi(\mathrm{d}x). \tag{1.1}$$

The Lévy process is of bounded variation, for which we use the notation $X \in bv$, if and only if $\sigma^2 = 0$ and $\int_{|x| \leq 1} |x| \Pi(\mathrm{d}x) < \infty$, and in that case, we denote by

$$\delta := \gamma - \int_{[-1,1]} x\Pi(\mathrm{d}x)$$

its drift coefficient.

We will continue the work of Blumenthal and Getoor [3] and Pruitt [13], in a sense, by investigating the possible limiting values taken by $t^{-\kappa} X_t$ as $t \downarrow 0$, where $\kappa > 0$ is some parameter. Recall that Blumenthal and Getoor [3] introduced the upper-index

$$\beta := \inf \left\{ \alpha > 0 : \int_{|x| \leq 1} |x|^\alpha \Pi(\mathrm{d}x) < \infty \right\} \in [0, 2],$$

which plays a critical role in this framework. Indeed, assuming for simplicity that the Brownian coefficient σ^2 is zero, and further that the drift coefficient δ is also 0 when $X \in bv$, then with probability one,

$$\limsup_{t \downarrow 0} \frac{|X_t|}{t^\kappa} = \begin{cases} 0 \\ \infty \end{cases} \text{according as} \begin{cases} \kappa < 1/\beta \\ \kappa > 1/\beta. \end{cases} \tag{1.2}$$

See also Pruitt [13]. Note that the critical case when $\kappa = 1/\beta$ is not covered by (1.2)

One application of this kind of study is to get information on the rate of growth of the process relative to power law functions, in both a one and a two-sided sense, at small times. More precisely, we are concerned with the values of $\limsup_{t \downarrow 0} |X_t|/t^\kappa$ and of $\limsup_{t \downarrow 0} X_t/t^\kappa$ (and the behaviour of

$\liminf_{t\downarrow 0} X_t/t^\kappa$ can be deduced from the limsup behaviour by performing a sign reversal). For example, when

$$\limsup_{t\downarrow 0} \frac{|X_t|}{t^\kappa} = \infty \text{ a.s.,} \tag{1.3}$$

then the regions $\{(t,y) \in [0,\infty)\times\mathbb{R} : |y| > at^\kappa\}$ are entered infinitely often for arbitrarily small t, a.s., for all $a > 0$. This can be thought of as a kind of "regularity" of X for these regions, at 0. We will refer to this kind of behaviour as crossing a "two-sided" boundary. On the other hand, when

$$\limsup_{t\downarrow 0} \frac{X_t}{t^\kappa} = \infty \text{ a.s.,} \tag{1.4}$$

we have "one-sided" (up)crossings; and similarly for downcrossings, phrased in terms of the liminf. In general, the process crosses one or two-sided boundaries in quite different ways, and, often (for many values of κ), when the limsups are finite a.s., they are in fact zero, a.s., as we will show. But the limsups may in some circumstances take finite, nonzero, values, a.s. Our aim here is to give necessary and sufficient conditions (NASC) which distinguish all these possibilities, for all values of $\kappa > 0$.

Let us eliminate at the outset certain cases which are trivial or easily deduced from known results. A result of Khintchine [10] (see Sato ([15], Prop. 47.11, p. 358)) is that, for any Lévy process X with Brownian coefficient $\sigma^2 \geq 0$, we have

$$\limsup_{t\downarrow 0} \frac{|X_t|}{\sqrt{2t\log|\log t|}} = \sigma \text{ a.s.} \tag{1.5}$$

Thus we immediately see that (1.3) and (1.4) cannot hold for $0 < \kappa < 1/2$; we always have $\lim_{t\downarrow 0} X_t/t^\kappa = 0$ a.s. in these cases. Of course, this also agrees with (1.2), since, always, $\beta \leq 2$. More precisely, recall the decomposition $X_t = \sigma B_t + X_t^{(0)}$, where $X^{(0)}$ is a Lévy process with characteristics $(\gamma, 0, \Pi)$ and B is an independent Brownian motion. Khintchine's law of the iterated logarithm for B and (1.5) applied for $X^{(0)}$ give

$$-\liminf_{t\downarrow 0} \frac{X_t}{\sqrt{2t\log|\log t|}} = \limsup_{t\downarrow 0} \frac{X_t}{\sqrt{2t\log|\log t|}} = \sigma \text{ a.s.} \tag{1.6}$$

So the one and two-sided limsup behaviours of X are precisely determined when $\sigma^2 > 0$ (regardless of the behaviour of $\Pi(\cdot)$, which may not even be

present). With these considerations, it is clear that *throughout, we can assume*

$$\sigma^2 = 0\,. \tag{1.7}$$

Furthermore, we can restrict attention to the cases $\kappa \geq 1/2$.

A result of Shtatland [16] and Rogozin [14] is that $X \notin bv$ if and only if

$$-\liminf_{t\downarrow 0} \frac{X_t}{t} = \limsup_{t\downarrow 0} \frac{X_t}{t} = \infty \text{ a.s.},$$

so (1.3) and (1.4) hold for all $\kappa \geq 1$, in this case (and similarly for the liminf). On the other hand, when $X \in bv$, we have

$$\lim_{t\downarrow 0} \frac{X_t}{t} = \delta, \text{ a.s.},$$

where δ is the drift of X (cf. [1], p.84). Thus if $\delta > 0$, (1.4) holds for all $\kappa > 1$, but for no $\kappa \leq 1$, while if $\delta < 0$, (1.4) can hold for no $\kappa > 0$; while (1.3) holds in either case, with $\kappa > 1$, but for no $\kappa \leq 1$. *Thus, when $X \in bv$, we need only consider the case $\delta = 0$.*

The main statements for two-sided (respectively, one-sided) boundary crossing will be given in Section 2 (respectively, Section 3) and proved in Section 4 (respectively, Section 5). We use throughout similar notation to [5], [6] and [7]. In particular, we write $\Pi^{\#}$ for the Lévy measure of $-X$, then $\Pi^{(+)}$ for the restriction of Π to $[0,\infty)$, $\Pi^{(-)}$ for the restriction of $\Pi^{\#}$ to $[0,\infty)$, and

$$\begin{aligned}
\overline{\Pi}^{(+)}(x) &= \Pi((x,\infty)),\\
\overline{\Pi}^{(-)}(x) &= \Pi((-\infty,-x))\,,\\
\overline{\Pi}(x) &= \overline{\Pi}^{(+)}(x) + \overline{\Pi}^{(-)}(x),\ x > 0,
\end{aligned} \tag{1.8}$$

for the tails of $\Pi(\cdot)$. Recall that we assume (1.7) and that the Lévy measure Π is restricted to $[-1,1]$. We will often make use of the Lévy-Itô decomposition, which can be written as

$$X_t = \gamma t + \int_{[0,t]\times[-1,1]} x N(\mathrm{d}s,\mathrm{d}x)\,, \qquad t \geq 0\,, \tag{1.9}$$

where $N(\mathrm{d}t,\mathrm{d}x)$ is a Poisson random measure on $\mathbb{R}_+ \times [-1,1]$ with intensity $\mathrm{d}t\Pi(\mathrm{d}x)$ and the Poissonian stochastic integral above is taken in the compensated sense. See Theorem 41 on page 31 in [12] for details.

2 Two-Sided Case

In this section we study two-sided crossings of power law boundaries at small times. We wish to find a necessary and sufficient condition for (1.3) for each value of $\kappa \ge 1/2$. This question is completely answered in the next two theorems, where the first can be viewed as a reinforcement of (1.2).

Theorem 2.1 *Assume* (1.7)*, and take* $\kappa > 1/2$. *When* $X \in bv$*, assume its drift is zero.*
(i) If

$$\int_0^1 \overline{\Pi}(x^\kappa)\mathrm{d}x < \infty, \tag{2.1}$$

then we have

$$\lim_{t\downarrow 0} \frac{X_t}{t^\kappa} = 0 \quad a.s. \tag{2.2}$$

(ii) Conversely, if (2.1) *fails, then*

$$\limsup_{t\downarrow 0} \frac{|X_t - a(t)|}{t^\kappa} = \infty \ a.s.,$$

for any deterministic function $a(t) : [0,\infty) \mapsto \mathbb{R}$.

Remark 1 *It is easy to check that* (2.1) *is equivalent to*

$$\int_{[-1,1]} |x|^{1/\kappa}\Pi(\mathrm{d}x) < \infty.$$

The latter holds for $0 < \kappa \le 1/2$ *for any Lévy process, as a fundamental property of the Lévy canonical measure ([1], p.13).* (2.2) *always holds when* $0 < \kappa < 1/2$*, as mentioned in Section 1, but not necessarily when* $\kappa = 1/2$.

The case $\kappa = 1/2$ which is excluded in Theorem 2.1 turns out to have interesting and unexpected features. To put these in context, let's first review some background. Khintchine [10] (see also Sato ([15], Prop. 47.12, p. 358)) showed that, given any function $h(t)$, positive, continuous, and nondecreasing in a neighbourhood of 0, and satisfying $h(t) = o\left(\sqrt{t\log|\log t|}\right)$ as $t \downarrow 0$, there is a Lévy process with $\sigma^2 = 0$ such that $\limsup_{t\downarrow 0} |X_t|/h(t) = \infty$ a.s. For example, we can take $h(t) = \sqrt{t}\,(\log|\log t|)^{1/4}$. The corresponding Lévy

process satisfies $\limsup_{t\downarrow 0}|X_t|/\sqrt{t} = \infty$ a.s. Thus the implication (2.1) $\Rightarrow$ (2.2) is not in general true when $\kappa = 1/2$.

On the other hand, when $\kappa = 1/2$, Theorem 2.1 remains true for example when $X \in bv$, in the sense that both (2.1) and (2.2) then hold, as follows from the fact that $X_t = O(t)$ a.s. as $t \downarrow 0$.

Thus we can have $\limsup_{t\downarrow 0}|X_t|/\sqrt{t}$ equal to 0 or ∞ a.s., and we are led to ask for a NASC to decide between the alternatives. We give such a condition in Theorem 2.2 and furthermore show that $\limsup_{t\downarrow 0}|X_t|/\sqrt{t}$ may take a positive finite value, a.s. Remarkably, Theorem 2.2 simultaneously solves the one-sided problem. These one sided cases are further investigated in Section 3, where it will be seen that, by contrast, the one and two sided situations are completely different when $\kappa \neq 1/2$.

To state the theorem, we need the notation

$$V(x) = \int_{|y|\le x} y^2\Pi(\mathrm{d}y), \ x > 0. \tag{2.3}$$

Theorem 2.2 *(The case $\kappa = 1/2$.) Assume* (1.7)*, and put*

$$I(a) = \int_0^1 x^{-1}\exp\left(-\frac{a^2}{2V(x)}\right)\mathrm{d}x$$

and

$$\lambda_I^* := \inf\{a > 0 : I(a) < \infty\} \in [0, \infty]$$

(with the convention, throughout, that the inf of the empty set is $+\infty$). Then, a.s.,

$$-\liminf_{t\downarrow 0}\frac{X_t}{\sqrt{t}} = \limsup_{t\downarrow 0}\frac{X_t}{\sqrt{t}} = \limsup_{t\downarrow 0}\frac{|X_t|}{\sqrt{t}} = \lambda_I^*. \tag{2.4}$$

Remark 2 *(i)* (2.4) *forms a nice counterpart to the iterated log version in* (1.5) *and* (1.6)*.*

(ii) If (2.1) *holds for some $\kappa > 1/2$, then $V(x) = o(x^{2-1/\kappa})$ as $x \downarrow 0$, so $\int_0^1 \exp\{-a^2/2V(x)\}\mathrm{d}x/x$ converges for all $\lambda > 0$. Thus $\lambda_I^* = 0$ and $\lim_{t\downarrow 0} X_t/\sqrt{t} = 0$ a.s. in this case, according to Theorem 2.2. Of course, this agrees with Theorem 2.1(i).*

(iii) The convergence of $\int_{|x|\le \mathrm{e}^{-e}} x^2\log|\log|x||\Pi(\mathrm{d}x)$ implies the convergence of $\int_0^1 \exp\{-a^2/2V(x)\}\mathrm{d}x/x$ for all $a > 0$, as is easily checked, hence

we have $\lim_{t\downarrow 0}|X_t|/\sqrt{t} = 0$ *a.s. for all such Lévy processes. A finite positive value, a.s., for* $\limsup_{t\downarrow 0}|X_t|/\sqrt{t}$ *can occur only in a small class of Lévy processes whose canonical measures have* $\Pi(\mathrm{d}x)$ *close to* $|x|^{-3}\mathrm{d}x$ *near* 0*. For example, we can find a* Π *such that, for small* x, $V(x) = 1/\log|\log x|$*. Then* $\int_0^{1/2}\exp\{-a^2/2V(x)\}\mathrm{d}x/x = \int_0^{1/2}|\log x|^{-a^2/2}\mathrm{d}x/x$ *is finite for* $a > \sqrt{2}$ *but infinite for* $a \le \sqrt{2}$*. Thus* $\limsup_{t\downarrow 0}|X_t|/\sqrt{t} = \sqrt{2}$ *a.s. for this process; in fact,* $\limsup_{t\downarrow 0}X_t/\sqrt{t} = \sqrt{2}$ *a.s., and* $\liminf_{t\downarrow 0}X_t/\sqrt{t} = -\sqrt{2}$ *a.s.*

(iv) Theorem 2.2 tells us that the only possible a.s. limit, as $t \downarrow 0$*, of* $X_t/\sqrt{t}$ *is 0, and that this occurs iff* $\lambda_I^* = 0$*, i.e., iff* $I(\lambda) < \infty$ *for all* $\lambda > 0$*. Similarly, the iterated log version in* (1.6) *gives that the only possible a.s. limit, as* $t \downarrow 0$*, of* $X_t/\sqrt{t\log|\log t|}$ *is 0, and that this occurs iff* $\sigma^2 = 0$*. When* $\kappa > 1/2$*, Theorem 2.1 gives that* $\lim_{t\downarrow 0}X_t/t^\kappa = 0$ *a.s. iff* $\int_0^1\overline{\Pi}(x^\kappa)\mathrm{d}x < \infty$*, provided, when* $\kappa \ge 1$*, the drift* $\delta = 0$*.*

Another result in this vein is that we can have $\lim_{t\downarrow 0}X_t/t^\kappa = \delta$ *a.s for a constant* δ *with* $0 < |\delta| < \infty$*, and* $\kappa > 0$*, iff* $\kappa = 1$*,* $X \in bv$*,* δ *is the drift, and* $\delta \ne 0$*.*

The following corollary shows that centering has no effect in the two-sided case.

Corollary 2.1 *Assume* (1.7)*, and, if* $X \in bv$*, assume it has drift zero. Suppose* $\limsup_{t\downarrow 0}|X_t|/t^\kappa = \infty$ *a.s., for some* $\kappa \ge 1/2$*. Then*

$$\limsup_{t\downarrow 0}\frac{|X_t - a(t)|}{t^\kappa} = \infty \text{ a.s., for any nonstochastic } a(t). \tag{2.5}$$

Finally, in this section, Table 1 summarises the conditions for (1.3):

Table 1

Value of κ	NASC for $\limsup_{t\downarrow 0}\|X_t\|/t^\kappa = \infty$ a.s. (when $\sigma^2 = 0$)
$0 \le \kappa < \frac{1}{2}$	Never true
$\kappa = \frac{1}{2}$	$\lambda_I^* = \infty$ (See Theorem 2.2)
$\frac{1}{2} < \kappa \le 1$	$\int_0^1\overline{\Pi}(x^\kappa)\mathrm{d}x = \infty$
$\kappa > 1$, $X \in bv$, $\delta = 0$	$\int_0^1\overline{\Pi}(x^\kappa)\mathrm{d}x = \infty$
$\kappa > 1$, $X \in bv$, $\delta \ne 0$	Always true
$\kappa > 1$, $X \notin bv$	Always true

3 One-Sided Case

We wish to test for the finiteness or otherwise of $\limsup_{t\downarrow 0} X_t/t^\kappa$, so we proceed by finding conditions for

$$\limsup_{t\downarrow 0} \frac{X_t}{t^\kappa} = +\infty \text{ a.s.} \tag{3.1}$$

In view of the discussion in Section 1, and the fact that the case $\kappa = 1/2$ is covered in Theorem 2.2, we have only two cases to consider:

(a) $X \notin bv$, $1/2 < \kappa < 1$;

(b) $X \in bv$, with drift $\delta = 0$, $\kappa > 1$.

For Case (a), we need to define, for $1 \geq y > 0$, and for $\lambda > 0$,

$$W(y) := \int_0^y \int_x^1 z\Pi^{(+)}(\mathrm{d}z)\mathrm{d}x,$$

and then

$$J(\lambda) := \int_0^1 \exp\left\{ -\lambda \left(\frac{y^{\frac{2\kappa-1}{\kappa}}}{W(y)} \right)^{\frac{\kappa}{1-\kappa}} \right\} \frac{\mathrm{d}y}{y}. \tag{3.2}$$

Also let $\lambda_J^* := \inf\{\lambda > 0 : J(\lambda) < \infty\}$.

Theorem 3.1 *Assume* (1.7) *and keep* $1/2 < \kappa < 1$. *Then* (3.1) *holds if and only if*

(i) $\int_0^1 \overline{\Pi}^{(+)}(x^\kappa)\mathrm{d}x = \infty$, *or*

(ii) $\int_0^1 \overline{\Pi}^{(+)}(x^\kappa)\mathrm{d}x < \infty = \int_0^1 \overline{\Pi}^{(-)}(x^\kappa)\mathrm{d}x$, *and* $\lambda_J^* = \infty$.

When (i) and (ii) fail, we have in greater detail: suppose

(iii) $\int_0^1 \overline{\Pi}(x^\kappa)\mathrm{d}x < \infty$, *or*

(iv) $\int_0^1 \overline{\Pi}^{(+)}(x^\kappa)\mathrm{d}x < \infty = \int_0^1 \overline{\Pi}^{(-)}(x^\kappa)\mathrm{d}x$ *and* $\lambda_J^* = 0$.

Then

$$\limsup_{t\downarrow 0} \frac{X_t}{t^\kappa} = 0 \text{ a.s.} \tag{3.3}$$

Alternatively, suppose

(v) $\int_0^1 \overline{\Pi}^{(+)}(x^\kappa)\mathrm{d}x < \infty = \int_0^1 \overline{\Pi}^{(-)}(x^\kappa)\mathrm{d}x$ *and* $\lambda_J^* \in (0,\infty)$. *Then*

$$\limsup_{t\downarrow 0} \frac{X_t}{t^\kappa} = c \text{ a.s., for some } c \in (0,\infty). \tag{3.4}$$

Remark 3 *(i) Using the integral criterion in terms of* $J(\lambda)$*, it's easy to give examples of all three possibilities* (0*,* ∞*, or in* $(0,\infty)$*) for* $\limsup_{t\downarrow 0} X_t/t^\kappa$*, in the situation of Theorem 3.1.*

(ii) Note that $X \notin bv$ *when* $\int_0^1 \overline{\Pi}^{(+)}(x^\kappa)\mathrm{d}x = \infty$ *or* $\int_0^1 \overline{\Pi}^{(-)}(x^\kappa)\mathrm{d}x = \infty$ *in Theorem 3.1, because* $\overline{\Pi}^{(\pm)}(x^\kappa) \le \overline{\Pi}^{(\pm)}(x)$ *when* $0 < x < 1$ *and* $\kappa < 1$*, so* $\int_0^1 \overline{\Pi}(x)\mathrm{d}x = \infty$*.*

(iii) It may seem puzzling at first that a second moment-like function, $V(\cdot)$*, appears in Theorem 2.2, whereas* $W(\cdot)$*, a kind of integrated first moment function, appears in Theorem 3.1. Though closely related, in general,* $V(x)$ *is not asymptotically equivalent to* $W(x)$*, as* $x \to 0$*, and neither function is asymptotically equivalent to yet another second moment-like function on* $[0,\infty)$*,* $U(x) := V(x) + x^2\overline{\Pi}(x)$*.* $V(x)$ *arises naturally in the proof of Theorem 2.2, which uses a normal approximation to certain probabilities, whereas* $W(x)$ *arises naturally in the proof of Theorem 3.1, which uses Laplace transforms and works with spectrally one-sided Lévy processes. It is possible to reconcile the different expressions; in fact, Theorem 2.2 remains true if* V *is replaced in the integral* $I(\lambda)$ *by* U *or by* W*. Thus these three functions are equivalent in the context of Theorem 2.2 (but not in general). We explain this in a little more detail following the proof of Theorem 3.1.*

Next we turn to Case (b). When $X \in bv$ we can define, for $0 < x < 1$,

$$A_+(x) = \int_0^x \overline{\Pi}^{(+)}(y)\mathrm{d}y \text{ and } A_-(x) = \int_0^x \overline{\Pi}^{(-)}(y)\mathrm{d}y. \tag{3.5}$$

Theorem 3.2 *Assume* (1.7)*, suppose* $\kappa > 1$*,* $X \in bv$*, and its drift* $\delta = 0$*. If*

$$\int_{(0,1]} \frac{\Pi^{(+)}(\mathrm{d}x)}{x^{-1/\kappa} + A_-(x)/x} = \infty \tag{3.6}$$

then (3.1) *holds. Conversely, if* (3.6) *fails, then* $\limsup_{t\downarrow 0} X_t/t^\kappa \le 0$ *a.s.*

Remark 4 *(i) It's natural to enquire whether* (3.6) *can be simplified by considering separately integrals containing the components of the integrand in* (3.6)*. This is not the case. For each* $\kappa > 1$*, it is possible to find a Lévy process* $X \in bv$ *with drift* 0 *for which* (3.6) *fails but*

$$\int_{(0,1]} x^{\frac{1}{\kappa}} \Pi^{(+)}(\mathrm{d}x) = \infty = \int_{(0,1]} (x/A_-(x))\, \Pi^{(+)}(\mathrm{d}x).$$

The idea is to construct a continuous increasing concave function which is linear on a sequence of intervals tending to 0*, which can serve as an* $A_-(x)$*, and which oscillates around the function* $x \mapsto x^{1-1/\kappa}$*. Note that* (3.6) *is equivalent to*

$$\int_{(0,1]} \min\left(x^{1/\kappa}, \frac{x}{A_-(x)}\right) \Pi^{(+)}(\mathrm{d}x) = \infty.$$

We will omit the details of the construction.

(ii) It is possible to have $\limsup_{t\downarrow 0} X_t/t^\kappa < 0$ *a.s., in the situation of Theorem 3.2, when* (3.6) *fails; for example, when* X *is the negative of a subordinator with zero drift. The value of the limsup can then be determined by applying Lemma 5.3 in Section 4.*

(ii) For another equivalence, we note that (3.6) *holds if and only if*

$$\int_0^1 \overline{\Pi}^{(+)}(t^\kappa + X_t^{(-)})dt = \infty \text{ a.s.} \tag{3.7}$$

where $X^{(-)}$ *is a subordinator with drift* 0 *and Lévy measure* $\Pi^{(-)}$*. This can be deduced from Erickson and Maller [8], Theorem 1, and provides a connection between the a.s. divergence of the Lévy integral in* (3.7) *and the upcrossing condition* (3.1).

Table 2 summarises the conditions for (3.1):

Table 2

Value of κ	NASC for $\limsup_{t\downarrow 0} X_t/t^\kappa = \infty$ a.s. (when $\sigma^2 = 0$)
$0 \le \kappa < \frac{1}{2}$	Never true
$\kappa = \frac{1}{2}$, $X \notin bv$	See Theorem 2.2
$\frac{1}{2} < \kappa < 1$, $X \notin bv$	See Theorem 3.1
$\frac{1}{2} \le \kappa \le 1$, $X \in bv$	Never true
$\kappa > 1$, $X \in bv$, $\delta < 0$	Never true
$\kappa > 1$, $X \in bv$, $\delta = 0$	See Theorem 3.2
$\kappa > 1$, $X \in bv$, $\delta > 0$	Always true
$\kappa \ge 1$, $X \notin bv$	Always true

Our final theorem applies the foregoing results to give a criterion for

$$\lim_{t \downarrow 0} \frac{X_t}{t^\kappa} = +\infty \text{ a.s.} \tag{3.8}$$

This is a stronger kind of divergence of the normed process to ∞, for small times. A straightforward analysis of cases, using our one and two-sided results, shows that (3.8) never occurs if $0 < \kappa \leq 1$, if $\kappa > 1$ and $X \notin bv$, or if $\kappa > 1$ and $X \in bv$ with negative drift. If $\kappa > 1$ and $X \in bv$ with positive drift, (3.8) always occurs. That leaves just one case to consider, in:

Theorem 3.3 *Assume* (1.7), *suppose* $\kappa > 1$, $X \in bv$, *and its drift* $\delta = 0$. *Then* (3.8) *holds iff*

$$K_X(d) := \int_0^1 \frac{\mathrm{d}y}{y} \exp\left\{ -d \frac{(A_+(y))^{\frac{\kappa}{\kappa-1}}}{y} \right\} < \infty, \text{ for all } d > 0, \tag{3.9}$$

and

$$\int_{(0,1]} \frac{x}{A_+(x)} \Pi^{(-)}(\mathrm{d}x) < \infty. \tag{3.10}$$

Concluding Remarks. It's natural to enquire about a one-sided version of Corollary 2.1: when is

$$\limsup_{t \downarrow 0} \frac{X_t - a(t)}{t^\kappa} < \infty \text{ a.s., for some nonstochastic } a(t)? \tag{3.11}$$

Phrased in such a general way the question is not interesting since we can always make $X_t = o(a(t))$ a.s as $t \downarrow 0$ by choosing $a(t)$ large enough by comparison with X_t (e.g., $a(t)$ such that $a(t)/\sqrt{t \log |\log t|} \to \infty$, as $t \downarrow 0$, will do, by (1.5)), so the limsup in (3.11) becomes negative. So we would need to restrict $a(t)$ in some way. Section 3 deals with the case $a(t) = 0$. Another choice is to take $a(t)$ as a natural centering function such as EX_t or as a median of X_t. However, in our small time situation, EX_t is essentially 0 or the drift of X, so we are led back to the case $a(t) = 0$ again (and similarly for the median). Of course there may be other interesting choices of $a(t)$ in some applications, and there is the wider issue of replacing t^κ by a more general norming function. Some of our results in Sections 4 and 5 address the latter, but we will not pursue these points further here.

4 Proofs for Section 2

4.1 Proof of Theorem 2.1

The proof relies on a pair of technical results which we will establish first. Recall the notation $V(x)$ in (2.3).

Proposition 4.1 *Let $b : \mathbb{R}_+ \to [0, \infty)$ be any non-decreasing function such that*

$$\int_0^1 \overline{\Pi}(b(x))\mathrm{d}x < \infty \quad \textit{and} \quad \int_0^1 V(b(x))b^{-2}(x)\mathrm{d}x < \infty.$$

Then

$$\limsup_{t \downarrow 0} \frac{|X_t - a(t)|}{b(4t)} \leq 1 \text{ a.s.,}$$

where

$$a(t) := \gamma t - \int_0^t \mathrm{d}s \int_{b(s)<|x|\leq 1} x\Pi(\mathrm{d}x), \ t \geq 0. \tag{4.1}$$

Proof of Proposition 4.1: Recall the Lévy-Itô decomposition (1.9). In this setting, it is convenient to introduce

$$X_t^{(1)} := \int_{[0,t]\times[0,1]} \mathbf{1}_{\{|x|\leq b(s)\}} xN(\mathrm{d}s, \mathrm{d}x)$$

and

$$X_t^{(2)} := \gamma t + \int_{[0,t]\times[0,1]} \mathbf{1}_{\{b(s)<|x|\leq 1\}} xN(\mathrm{d}s, \mathrm{d}x)\,,$$

where again the stochastic integrals are taken in the compensated sense. Plainly, $X = X^{(1)} + X^{(2)}$.

The assumption $\int_0^1 \overline{\Pi}(b(x))\mathrm{d}x < \infty$ implies that

$$N\left(\{(s, x) : 0 \leq s \leq t \text{ and } b(s) < |x| \leq 1\}\right) = 0$$

whenever $t > 0$ is sufficiently small a.s., and in this situation $X^{(2)}$ is just γt minus the compensator, a.s.; i.e.,

$$X_t^{(2)} = \gamma t - \int_0^t \mathrm{d}s \int_{b(s)<|x|\leq 1} x\Pi(\mathrm{d}x) = a(t).$$

On the one hand, $X^{(1)}$ is a square-integrable martingale with oblique bracket

$$\langle X^{(1)}\rangle_t = \int_0^t \mathrm{d}s \int_{|x|\le b(s)} x^2 \Pi(\mathrm{d}x) = \int_0^t V(b(s))\mathrm{d}s \le tV(b(t)).$$

By Doob's maximal inequality, we have for every $t \ge 0$

$$P(\sup_{0\le s\le t} |X_s^{(1)}| > b(2t)) \le 4tV(b(t))b^{-2}(2t).$$

On the other hand, the assumptions that $b(t)$ is non-decreasing and that $\int_0^1 \mathrm{d}x V(b(x))b^{-2}(x) < \infty$ entail

$$\sum_{n=1}^{\infty} 2^{-n} V(b(2^{-n}))b^{-2}(2^{-n+1}) < \infty.$$

By the Borel-Cantelli lemma, we thus see that

$$\lim_{n\to\infty} \frac{\sup_{0\le s\le 2^{-n}} |X_s^{(1)}|}{b(2^{-n+1})} \le 1 \qquad \text{a.s.},$$

and the proof is completed by a standard argument of monotonicity. ∎

Proposition 4.2 *Suppose there are deterministic functions* $a : \mathbb{R}_+ \to \mathbb{R}$ *and* $b : (0,\infty) \to (0,\infty)$, *with* b *measurable, such that*

$$P\left(\limsup_{t\downarrow 0} \frac{|X_t - a(t)|}{b(t)} < \infty\right) > 0. \tag{4.2}$$

Then there is some finite constant C *such that*

$$\int_0^1 \overline{\Pi}(Cb(x))\mathrm{d}x < \infty. \tag{4.3}$$

Proof of Proposition 4.2: Symmetrise X by subtracting an independent equally distributed X' to get $X_t^{(s)} = X_t - X'_t$, $t \ge 0$. Then (4.2) and Blumenthal's 0-1 law imply there is some finite constant C such that

$$\limsup_{t\downarrow 0} \frac{|X_t^{(s)}|}{b(t)} < \frac{C}{2}, \text{ a.s.} \tag{4.4}$$

Suppose now that (4.3) fails. Note that $\overline{\Pi}^{(s)}(\cdot) = 2\overline{\Pi}(\cdot)$, where $\Pi^{(s)}$ is the Lévy measure of $X^{(s)}$, so that $\int_0^1 \overline{\Pi}^{(s)}(Cb(x))\mathrm{d}x = \infty$. Then from the Lévy-Itô decomposition, we have that, for every $\varepsilon > 0$,

$$\#\{t \in (0,\varepsilon] : |\Delta_t^{(s)}| > Cb(t)\} = \infty, \text{ a.s.},$$

where $\Delta_t^{(s)} = X_t^{(s)} - X_{t-}^{(s)}$. But whenever $|\Delta_t^{(s)}| > Cb(t)$, we must have $|X_{t-}^{(s)}| > Cb(t)/2$ or $|X_t^{(s)}| > Cb(t)/2$; which contradicts (4.4). Thus (4.3) holds. ∎

Finally, we will need an easy deterministic bound.

Lemma 4.1 *Fix some $\kappa \geq 1/2$ and assume*

$$\int_{|x|<1} |x|^{1/\kappa}\Pi(\mathrm{d}x) < \infty. \tag{4.5}$$

When $\kappa \geq 1$, $X \in bv$ and we suppose further that the drift coefficient $\delta = \gamma - \int_{|x|\leq 1} x\Pi(\mathrm{d}x)$ is 0. Then, as $t \to 0$,

$$a(t) = \gamma t - \int_0^t \mathrm{d}s \int_{s^\kappa < |x| \leq 1} x\Pi(\mathrm{d}x) = o(t^\kappa).$$

Proof of Lemma 4.1: Suppose first $\kappa < 1$. For every $0 < \varepsilon < \eta < 1$, we have

$$\begin{aligned}\int_{\varepsilon<|x|\leq 1} |x|\Pi(\mathrm{d}x) &\leq \varepsilon^{1-1/\kappa}\int_{|x|\leq\eta} |x|^{1/\kappa}\Pi(\mathrm{d}x) + \int_{\eta<|x|\leq 1}|x|\Pi(\mathrm{d}x) \\ &= \varepsilon^{1-1/\kappa} o_\eta + c(\eta), \text{ say,}\end{aligned}$$

where, by (4.5), $\lim_{\eta\downarrow 0} o_\eta = 0$. Since $\kappa < 1$, it follows that

$$\limsup_{t\downarrow 0} |a(t)|t^{-\kappa} \leq \kappa^{-1} o_\eta,$$

and as we can take η arbitrarily small, we conclude that $a(t) = o(t^\kappa)$.

In the case $\kappa \geq 1$, X has bounded variation with zero drift coefficient. We may rewrite $a(t)$ in the form

$$a(t) = \int_0^t \mathrm{d}s \int_{|x|\leq s^\kappa} x\Pi(\mathrm{d}x).$$

The assumption (4.5) entails $\int_{|x|\le\varepsilon}|x|\Pi(\mathrm{d}x) = o(\varepsilon^{1-1/\kappa})$ and we again conclude that $a(t) = o(t^\kappa)$. ∎

We now have all the ingredients to establish Theorem 2.1.

Proof of Theorem 2.1: Keep $\kappa > 1/2$ throughout. (i) Suppose (2.1) holds, which is equivalent to (4.5). Writing $|x|^{1/\kappa} = |x|^{1/\kappa-2}x^2$, we see from an integration by parts (Fubini's theorem) that $\int_0^1 V(x)x^{1/\kappa-3}\mathrm{d}x < \infty$. Note that the assumption that $\kappa \neq 1/2$ is crucial in this step. The change of variables $x = y^\kappa$ now gives that $\int_0^1 V(y^\kappa)y^{-2\kappa}\mathrm{d}y < \infty$. We may thus apply Proposition 4.1 and get that

$$\limsup_{t\downarrow 0}\frac{|X_t - a(t)|}{t^\kappa} \le 4^\kappa \qquad \text{a.s.}$$

where $a(t)$ is as in Lemma 4.1. We thus have shown that when (2.1) holds,

$$\limsup_{t\downarrow 0}\frac{|X_t|}{t^\kappa} \le 4^\kappa \qquad \text{a.s.}$$

For every $c > 0$, the time-changed process X_{ct} is a Lévy process with Lévy measure $c\Pi$, so we also have $\limsup_{t\downarrow 0}|X_{ct}|t^{-\kappa} \le 4^\kappa$ a.s. As we may take c as large as we wish, we conclude that

$$\lim_{t\downarrow 0}\frac{|X_t|}{t^\kappa} = 0 \qquad \text{a.s.}$$

(ii) By Proposition 4.2, if

$$P\left(\limsup_{t\downarrow 0}\frac{|X_t - a(t)|}{b(t)} < \infty\right) > 0,$$

then $\int_0^1 \overline{\Pi}(Cx^\kappa)\mathrm{d}x < \infty$ for some finite constant C. By an obvious change of variables, this shows that (2.1) must hold. This completes the proof of Theorem 2.1. ∎

Finally we establish Corollary 2.1.

Proof of Corollary 2.1: This hinges on the fact that when (2.5) fails, then $(X_t - a(t))/t^\kappa \xrightarrow{P} 0$, with $a(t)$ defined as in (4.1) – even for $\kappa = 1/2$. We will omit the details. ∎

4.2 Proof of Theorem 2.2

We now turn our attention to Theorem 2.2 and develop some notation and material in this direction. Write, for $b > 0$,

$$X_t = Y_t^{(b)} + Z_t^{(b)}, \tag{4.6}$$

with

$$\begin{aligned} Y_t^{(b)} &:= \int_{[0,t]\times[-1,1]} \mathbf{1}_{\{|x|\le b\}} x N(\mathrm{d}s, \mathrm{d}x)\,, \\ Z_t^{(b)} &:= \gamma t + \int_{[0,t]\times[-1,1]} \mathbf{1}_{\{b<|x|\}} x N(\mathrm{d}s, \mathrm{d}x)\,, \end{aligned} \tag{4.7}$$

where $N(\mathrm{d}s, \mathrm{d}x)$ is a Poisson random measure on $[0,\infty) \times [-1, 1]$ with intensity $\mathrm{d}s\Pi(\mathrm{d}x)$, and the stochastic integrals are taken in the compensated sense.

Lemma 4.2 *(No assumptions on X.) For every $0 < r < 1$ and $\varepsilon > 0$, we have*

$$\sum_{n=1}^{\infty} P(\sup_{0\le t\le r^n} |Z_t^{(r^{n/2})}| > \varepsilon r^{n/2}) < \infty,$$

and as a consequence,

$$\lim_{n\to\infty} r^{-n/2} \sup_{0\le t\le r^n} |Z_t^{(r^{n/2})}| = 0 \text{ a.s.}$$

Proof of Lemma 4.2: Introduce, for every integer n, the set

$$A_n := [0, r^n] \times ([-1, -r^{n/2}) \cup (r^{n/2}, 1])\,,$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} P(N(A_n) > 0) &\le \sum_{n=1}^{\infty} r^n \overline{\Pi}(r^{n/2}) \\ &\le (1-r)^{-1} \sum_{n=1}^{\infty} \int_{r^{n+1}}^{r^n} \overline{\Pi}(\sqrt{x})\mathrm{d}x \\ &\le (1-r)^{-1} \int_0^1 \overline{\Pi}(\sqrt{x})\mathrm{d}x. \end{aligned}$$

As the last integral is finite (always), we have from the Borel-Cantelli lemma that $N(A_n) = 0$ whenever n is sufficiently large, a.s.

On the other hand, on the event $N(A_n) = 0$, we have

$$Z_t^{(r^{n/2})} = t\left(\gamma - \int_{r^{n/2}<|x|\le 1} x\Pi(\mathrm{d}x)\right), \qquad 0 \le t \le r^n.$$

Again as a result of the convergence of $\int_{|x|\le 1} x^2\Pi(\mathrm{d}x)$, the argument in Lemma 4.1 shows that the supremum over $0 \le t \le r^n$ of the absolute value of the right-hand side is $o(r^{n/2})$. The Borel-Cantelli lemma completes the proof. ∎

In view of Lemma 4.2 we can concentrate on $Y_t^{(\sqrt{t})}$ in (4.6). We next prove:

Lemma 4.3 *Let Y be a Lévy process with canonical measure Π_Y, satisfying $EY_1 = 0$ and $m_4 < \infty$, where $m_k := \int_{x\in\mathbb{R}} |x|^k \Pi_Y(\mathrm{d}x)$, $k = 2, 3, \cdots$.*
(i) Then

$$\lim_{t\downarrow 0} \frac{1}{t} E|Y_t|^3 = m_3.$$

(ii) For any $x > 0$, $t > 0$, we have the bound

$$\left|P(Y_t > x\sqrt{tm_2}) - \overline{F}(x)\right| \le \frac{Am_3}{\sqrt{t}m_2^{3/2}(1+x)^3}, \tag{4.8}$$

where

$$\overline{F}(x) = \int_x^\infty \mathrm{e}^{-y^2/2}\mathrm{d}y/\sqrt{2\pi} = \frac{1}{2}\mathrm{erfc}(x/\sqrt{2})$$

is the tail of the standard normal distribution function, and A is an absolute constant.

Proof of Lemma 4.3: (i) We can calculate $EY_t^4 = tm_4 + 3t^2m_2^2$. So by Chebychev's inequality for second and fourth moments, for $x > 0$, $t > 0$,

$$\frac{1}{t}P(|Y_t| > x) \le \frac{m_2}{x^2}\mathbf{1}_{\{0<x\le 1\}} + \frac{m_4 + 3tm_2^2}{x^4}\mathbf{1}_{\{x>1\}}.$$

We can also calculate

$$\frac{1}{t}E|Y_t|^3 = \frac{3}{t}\int_0^\infty x^2 P(|Y_t| > x)\mathrm{d}x.$$

By [1], Ex. 1, p. 39, $P(|Y_t| > x)/t \to \overline{\Pi}_Y(x)$, as $t \downarrow 0$, for each $x > 0$. The result (i) follows by dominated convergence.

(ii) Write $Y_t = \sum_{i=1}^n Y(i,t)$, for $n = 1, 2, \cdots$, where $Y(i,t) := Y(it/n) - Y((i-1)t/n)$ are i.i.d., each with the distribution of $Y(t/n)$. According to a non-uniform Berry-Esseen bound (Theorem 14, p.125 of Petrov [11]), for each $n = 1, 2, \cdots$, (4.8) holds with the righthand side replaced by

$$\frac{AE|Y(t/n)|^3}{\sqrt{n}(tm_2/n)^{3/2}(1+x)^3} = \frac{AE|Y(t/n)|^3/(t/n)}{\sqrt{t}m_2^{3/2}(1+x)^3}.$$

By Part (i) this tends as $n \to \infty$ to the righthand side of (4.8). ∎

Proposition 4.3 *In the notation* (4.7)*, we have, for* $a > 0$, $0 < r < 1$,

$$\sum_{n\geq 0} P\left(Y_{r^n}^{(r^{n/2})} > ar^{n/2}\right) < \infty \Longleftrightarrow \int_0^1 \sqrt{V(x)} \exp\left(\frac{-a^2}{2V(x)}\right) \frac{\mathrm{d}x}{x} < \infty. \tag{4.9}$$

Proof of Proposition 4.3: For every fixed $t > 0$, $Y_s^{(\sqrt{t})}$ is the compensated sum of jumps of X smaller in magnitude than $\sqrt{t}$, up to time s. It is a centered Lévy process with canonical measure $\mathbf{1}_{\{|x|\leq\sqrt{t}\}}\Pi(\mathrm{d}x)$, $x \in \mathbb{R}$. Applying Lemma 4.3, we get $m_2 = V(\sqrt{t})$ and $m_3 = \int_{|y|\leq\sqrt{t}} |y|^3\Pi(\mathrm{d}y) = \rho(\sqrt{t})$, say. Then we get, for $x > 0$,

$$|P\left(Y_t^{(\sqrt{t})} > x\sqrt{tV(\sqrt{t})}\right) - \overline{F}(x)| \leq \frac{A\rho(\sqrt{t})}{\sqrt{tV^3(\sqrt{t})}(1+x)^3}.$$

Replacing x by $a/\sqrt{V(\sqrt{t})}$, $a > 0$, we have

$$\left|P\left(Y_t^{(\sqrt{t})} > a\sqrt{t}\right) - \overline{F}\left(a/\sqrt{V(\sqrt{t})}\right)\right| \leq \varepsilon(t) := \frac{A\rho(\sqrt{t})}{\sqrt{t}a^3},$$

and we claim that $\sum \varepsilon(r^n) < \infty$. In fact, for some $c > 0$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\rho(r^{n/2})}{r^{n/2}} &= \sum_{n=0}^{\infty} \frac{1}{r^{n/2}} \sum_{j \ge n} \int_{r^{(j+1)/2} < |y| \le r^{j/2}} |y|^3 \Pi(\mathrm{d}y) \\
&= \sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} r^{-n/2} \right) \int_{r^{(j+1)/2} < |y| \le r^{j/2}} |y|^3 \Pi(\mathrm{d}y) \\
&\le c \sum_{j=0}^{\infty} r^{-j/2} \int_{r^{(j+1)/2} < |y| \le r^{j/2}} |y|^3 \Pi(\mathrm{d}y) \\
&\le c \sum_{j=0}^{\infty} \int_{r^{(j+1)/2} < |y| \le r^{j/2}} y^2 \Pi(\mathrm{d}y) \\
&= c \int_{|y| \le 1} y^2 \Pi(\mathrm{d}y) < \infty.
\end{aligned}$$

The result (4.9) follows, since the monotonicity of $\overline{F}$ shows that the convergence of $\sum_{n \ge 1} \overline{F}\left(a/\sqrt{V(r^{n/2})}\right)$ is equivalent to that of

$$\int_0^1 \overline{F}\left(a/\sqrt{V(\sqrt{x})}\right) \frac{\mathrm{d}x}{x} = \sum \int_{r^{\frac{n+1}{2}}}^{r^{\frac{n}{2}}} \overline{F}\left(a/\sqrt{V(\sqrt{x})}\right) \frac{\mathrm{d}x}{x},$$

and it is well-known that $\overline{F}(x) \backsim (2\pi)^{-1/2} x^{-1} \mathrm{e}^{-x^2/2}$ as $x \to \infty$. ∎

We can now establish Theorem 2.2.

Proof of Theorem 2.2: Recall the definition of $I(\cdot)$ in the statement of Theorem 2.2. We will first show that for every given $a > 0$

$$I(a) < \infty \quad \Rightarrow \limsup_{t \downarrow 0} \frac{X_t}{\sqrt{t}} \le a \text{ a.s.} \tag{4.10}$$

To see this, observe when $I(a) < \infty$, the integral in (4.9) converges, hence so does the series. Use the maximal inequality in Theorem 12, p.50 of Petrov

[11], to get, for $t>0$, $b>0$, $x>0$,

$$
\begin{aligned}
P\left(\sup_{0<s\le t} Y_s^{(b)} > x\right) &= \lim_{k\to\infty} P\left(\max_{1\le j\le \lceil kt\rceil} Y_{j/k}^{(b)} > x\right) \\
&= \lim_{k\to\infty} P\left(\max_{1\le j\le \lceil kt\rceil} \sum_{i=1}^{j} \Delta(i,k,b) > x\right) \\
&\le \limsup_{k\to\infty} 2P\left(\sum_{i=1}^{\lceil kt\rceil} \Delta(i,k,b) > x - \sqrt{2ktV(b)/k}\right) \\
&= 2P\left(Y_t^{(b)} > x - \sqrt{2tV(b)}\right), \qquad (4.11)
\end{aligned}
$$

where we note that $\left(\Delta(i,k,b) := Y_{i/k}^{(b)} - Y_{(i-1)/k}^{(b)}\right)_{i\ge 1}$ are i.i.d., each with expectation 0 and variance equal to $V(b)/k$. Given $\varepsilon>0$, replace t by r^n, b by $r^{n/2}$, and x by $ar^{n/2}+\sqrt{2r^nV(r^{n/2})}$, which is not larger than $(a+\varepsilon)r^{n/2}$, once n is large enough, in (4.11). The convergence of the series in (4.9) then gives

$$
\sum_{n\ge 0} P\left(\sup_{0<s\le r^n} Y_s^{(r^{n/2})} > (a+\varepsilon)r^{n/2}\right) < \infty \text{ for all } \varepsilon>0.
$$

Hence by the Borel-Cantelli Lemma

$$
\limsup_{n\to\infty} \frac{\sup_{0<t\le r^n} Y_t^{(r^{n/2})}}{r^{n/2}} \le a, \text{ a.s.} \qquad (4.12)
$$

Using (4.6), together with Lemma 4.2 and (4.12), gives

$$
\limsup_{n\to\infty} \frac{\sup_{0<t\le r^n} X_t}{r^{n/2}} \le a, \text{ a.s.}
$$

By an argument of monotonicity, this yields

$$
\limsup_{t\downarrow 0} \frac{X_t}{\sqrt{t}} \le \frac{a}{\sqrt{r}}, \text{ a.s.}
$$

Then let $r\uparrow 1$ to get $\limsup_{t\downarrow 0} X_t/\sqrt{t} \le a$ a.s.

For a reverse inequality, we show that for every $a>0$,

$$
I(a)=\infty \;\Rightarrow\; \limsup_{t\downarrow 0} \frac{X_t}{\sqrt{t}} \ge a, \text{ a.s.} \qquad (4.13)
$$

To see this, suppose that $I(a) = \infty$ for a given $a > 0$. Then the integral in (4.9) diverges when a is replaced by $a - \varepsilon$ for an arbitrarily small $\varepsilon > 0$, because $V(x) \geq \varepsilon \exp(-\varepsilon/2V(x))/2$, for $\varepsilon > 0$, $x > 0$. Hence, keeping in mind (4.6), Lemma 4.2 and Proposition 4.3, we deduce

$$\sum_{n \geq 0} P\left(X_{r^n} > a' r^{n/2}\right) = \infty \tag{4.14}$$

for all $a' < a$. For a given $\varepsilon > 0$, define for every integer $n \geq 0$ the events

$$\begin{aligned} A_n &= \{X_{r^n/(1-r)} - X_{r^{n+1}/(1-r)} > a' r^{n/2}\}, \\ B_n &= \{|X_{r^{n+1}/(1-r)}| \leq \varepsilon r^{n/2}\}. \end{aligned}$$

Then the $\{A_n\}_{n \geq 0}$ are independent, and each B_n is independent of the collection $\{A_n, A_{n-1}, \cdots, A_0\}$. Further, $\sum_{n \geq 0} P(A_n) = \infty$ by (4.14), so $P(A_n \text{ i.o.}) = 1$. It can be deduced easily from [1], Prop. 2(i), p.16, that $X_t/\sqrt{t} \xrightarrow{P} 0$, as $t \downarrow 0$, since (1.7) is enforced. Thus $P(B_n) \to 1$ as $n \to \infty$, and then, by the Feller–Chung lemma ([4], p. 69) we can deduce that $P(A_n \cap B_n \text{ i.o.}) = 1$. This implies $P(X_{r^n/(1-r)} > (a' - \varepsilon) r^{n/2} \text{ i.o.}) = 1$, thus

$$\limsup_{t \downarrow 0} \frac{X_t}{\sqrt{t}} \geq (a' - \varepsilon)\sqrt{1 - r} \text{ a.s.},$$

in which we can let $a' \uparrow a$, $\varepsilon \downarrow 0$ and $r \downarrow 0$ to get (4.13).

As just mentioned, we have $X_t/\sqrt{t} \xrightarrow{P} 0$, as $t \downarrow 0$, so $\liminf_{t \downarrow 0} X_t/\sqrt{t} \leq 0 \leq \limsup_{t \downarrow 0} X_t/\sqrt{t}$ a.s. Together with (4.10) and (4.13), this gives the statements in Theorem 2.2 (replace X by $-X$ to deduce the liminf statements from the limsup, noting that this leaves $V(\cdot)$ unchanged). ∎

5 Proofs for Section 3

5.1 Proof of Theorem 3.1

We start with some notation and technical results. Recall we assume (1.7).

Take $0 < \kappa < 1$ and suppose first

$$\int_0^1 \overline{\Pi}^{(+)}(x^\kappa) \mathrm{d}x = \infty. \tag{5.1}$$

Define, for $0 < x < 1$,

$$\rho_\kappa(x) = \frac{1}{\kappa}\int_x^1 y^{\frac{1}{\kappa}-1}\overline{\Pi}^{(+)}(y)\mathrm{d}y = \int_{x^{\frac{1}{\kappa}}}^1 \overline{\Pi}^{(+)}(y^\kappa)\mathrm{d}y.$$

Since $\overline{\Pi}^{(+)}(x) > 0$ for all small x, $\rho_\kappa(x)$ is strictly decreasing in a neighbourhood of 0, thus $x^{-1/\kappa}\rho_\kappa(x)$ is also strictly decreasing in a neighbourhood of 0, and tends to ∞ as $x \downarrow 0$ (because of (5.1)). Also define

$$U_+(x) = 2\int_0^x y\overline{\Pi}^{(+)}(y)\mathrm{d}y.$$

By a similar argument, $x^{-2}U_+(x)$ is strictly decreasing in a neighbourhood of 0, and tends to ∞ as $x \downarrow 0$.

Next, given $\alpha \in (0, \kappa)$, define, for $t > 0$,

$$c(t) = \inf\left\{x > 0 : \rho_\kappa(x)x^{-1/\kappa} + x^{-2}U_+(x) \leq \alpha/t\right\}.$$

Then $0 < c(t) < \infty$ for $t > 0$, $c(t)$ is strictly increasing, $\lim_{t\downarrow 0} c(t) = 0$, and

$$\frac{t\rho_\kappa(c(t))}{c^{\frac{1}{\kappa}}(t)} + \frac{tU_+(c(t))}{c^2(t)} = \alpha. \tag{5.2}$$

Since $\lim_{t\downarrow 0}\rho_\kappa(t) = \infty$, we have $\lim_{t\downarrow 0} c(t)/t^\kappa = \infty$.

We now point out that (5.1) can be reinforced as follows.

Lemma 5.1 *The condition* (5.1) *implies that*

$$\int_0^1 \overline{\Pi}^{(+)}(c(t))\mathrm{d}t = \infty.$$

Proof of Lemma 5.1: We will first establish

$$\int_0^1 \frac{\Pi(\mathrm{d}x)}{x^{-1/\kappa}\rho_\kappa(x) + x^{-2}U_+(x)} = \infty. \tag{5.3}$$

Suppose (5.3) fails. Since

$$f(x) := \frac{1}{x^{-1/\kappa}\rho_\kappa(x) + x^{-2}U_+(x)}$$

is nondecreasing (in fact, strictly increasing in a neighbourhood of 0) with $f(0) = 0$, for every $\varepsilon > 0$ there is an $\eta > 0$ such that, for all $0 < x < \eta$,

$$\varepsilon \geq \int_x^\eta f(z)\Pi(\mathrm{d}z) \geq f(x)\left(\overline{\Pi}^{(+)}(x) - \overline{\Pi}^{(+)}(\eta)\right),$$

giving

$$f(x)\overline{\Pi}^{(+)}(x) \leq \varepsilon + f(x)\overline{\Pi}^{(+)}(\eta) = \varepsilon + o(1), \text{ as } x \downarrow 0.$$

Letting $\varepsilon \downarrow 0$ shows that

$$\lim_{x\downarrow 0} f(x)\overline{\Pi}^{(+)}(x) = \lim_{x\downarrow 0}\left(\frac{\overline{\Pi}^{(+)}(x)}{x^{-1/\kappa}\rho_\kappa(x) + x^{-2}U_+(x)}\right) = 0. \tag{5.4}$$

It can be proved as in Lemma 4 of [6] that this implies

$$\lim_{x\downarrow 0}\left(\frac{x^{-1/\kappa}\rho_\kappa(x)}{x^{-2}U_+(x)}\right) = 0 \quad \text{or} \quad \liminf_{x\downarrow 0}\left(\frac{x^{-1/\kappa}\rho_\kappa(x)}{x^{-2}U_+(x)}\right) > 0. \tag{5.5}$$

Then, since (5.3) has been assumed not to hold,

$$\int_0^{1/2} \frac{x^{\frac{1}{\kappa}}}{\rho_\kappa(x)}\Pi(\mathrm{d}x) < \infty \quad \text{or} \quad \int_0^{1/2} \frac{x^2}{U_+(x)}\Pi(\mathrm{d}x) < \infty. \tag{5.6}$$

Noting that, under (5.1),

$$\begin{aligned}\rho_\kappa(x) &= \overline{\Pi}^{(+)}(1) - x^{\frac{1}{\kappa}}\overline{\Pi}^{(+)}(x) + \int_x^1 y^{\frac{1}{\kappa}}\Pi(\mathrm{d}y) \\ &\leq \overline{\Pi}^{(+)}(1) + \int_x^1 y^{\frac{1}{\kappa}}\Pi(\mathrm{d}y) \sim \int_x^1 y^{\frac{1}{\kappa}}\Pi(\mathrm{d}y), \text{ as } x \to 0,\end{aligned}$$

we see that the first relation in (5.6) is impossible because it would imply the finiteness of

$$\int_0^{1/2} x^{\frac{1}{\kappa}}\left(\int_x^1 y^{\frac{1}{\kappa}}\Pi(\mathrm{d}y)\right)^{-1}\Pi(\mathrm{d}x);$$

but this is infinite by (5.1) and the Abel-Dini theorem. In a similar way, the second relation in (5.6) can be shown to be impossible. Thus (5.3) is proved.

Then note that the inverse function $c^{\leftarrow}$ of c exists and satisfies, by (5.2),

$$c^{\leftarrow}(x) = \frac{\alpha}{x^{-1/\kappa}\rho_\kappa(x) + x^{-2}U_+(x)}.$$

Thus, by (5.3),

$$\int_0^1 c^{\leftarrow}(x)\Pi(\mathrm{d}x) = \infty = \int_0^{c^{\leftarrow}(1)} \overline{\Pi}^{(+)}(c(x))\mathrm{d}x. \tag{5.7}$$

This proves our claim. ∎

Proposition 5.1 *For every $\kappa < 1$, (5.1) implies*

$$\limsup_{t\downarrow 0} \frac{X_t}{t^\kappa} = \infty \qquad a.s.$$

Proof of Proposition 5.1: The argument relies on the analysis of the completely asymmetric case when the Lévy measure Π has support in $[0,1]$ or in $[-1,0]$. Since $\kappa < 1$, we can assume $\gamma = 0$ without loss of generality, because of course $\gamma t = o(t^\kappa)$. The Lévy-Itô decomposition (1.9) then yields

$$X_t = \widehat{X}_t + \widetilde{X}_t \tag{5.8}$$

with

$$\widehat{X}_t = \int_{[0,t]\times[0,1]} xN(\mathrm{d}s,\mathrm{d}x) \text{ and } \widetilde{X}_t = \int_{[0,t]\times[-1,0]} xN(\mathrm{d}s,\mathrm{d}x)\,, \tag{5.9}$$

where, as usual, the Poissonian stochastic integrals are taken in the compensated sense.

Choose α so small that

$$(1+\kappa)\alpha < 1/2 \text{ and } \alpha/(1/2-(1+\kappa)\alpha)^2 \le 1/2,$$

and then ε so small that $c(\varepsilon) < 1$. Observe that for every $0 < t < \varepsilon$

$$\begin{aligned} t\int_{\{c(t)<x\le 1\}} x\Pi(\mathrm{d}x) &= tc(t)\overline{\Pi}^{(+)}(c(t)) + t\lambda(c(t)) \\ &\le \frac{tU_+(c(t))}{c(t)} + t\lambda(c(t)) \\ &\le \alpha(1+\kappa)c(t), \end{aligned} \tag{5.10}$$

where the last inequality stems from (5.2) and

$$\lambda(x) := \int_x^1 \overline{\Pi}^{(+)}(y)\mathrm{d}y = \int_x^1 y^{1-\frac{1}{\kappa}} y^{\frac{1}{\kappa}-1}\overline{\Pi}^{(+)}(y)\mathrm{d}y \le \kappa x^{1-\frac{1}{\kappa}}\rho_\kappa(x)$$

(since $\kappa < 1$), so

$$\frac{t\lambda(c(t))}{c(t)} \leq \frac{\kappa t \rho_\kappa(c(t))}{c^{\frac{1}{\kappa}}(t)} \leq \kappa\alpha \quad \text{(by (5.2))}.$$

We next deduce from Lemma 5.1 that for every $\varepsilon > 0$, the Poisson random measure N has infinitely many atoms in the domain $\{(t,x) : 0 \leq t < \varepsilon \text{ and } x > c(t)\}$, a.s. Introduce

$$t_\varepsilon := \sup\{t \leq \varepsilon : N(\{t\} \times (c(t), 1]) = 1\},$$

the largest instant less than ε of such an atom. Our goal is to check that

$$P(X_{t_\varepsilon -} \geq -c(t_\varepsilon)/2) \geq 1/33 \tag{5.11}$$

for every $\varepsilon > 0$ sufficiently small, so that $P(X_{t_\varepsilon} > c(t_\varepsilon)/2) > 1/33$. Since $t^\kappa = o(c(t))$, it follows that for every $a > 0$

$$P(\exists t \leq \varepsilon : X_t > at^\kappa) \geq 1/33\,,$$

and hence $\limsup_{t\downarrow 0} X_t/t^\kappa = \infty$ with probability at least $1/33$. The proof is completed by an appeal to Blumenthal's 0-1 law.

In order to establish (5.11), we will work henceforth conditionally on t_ε; recall from the Markov property of Poisson random measures that the restriction of $N(\mathrm{d}t, \mathrm{d}x)$ to $[0, t_\varepsilon) \times [-1, 1]$ is still a Poisson random measure with intensity $\mathrm{d}t\Pi(\mathrm{d}x)$.

Recalling (5.10) and discarding the jumps $\widehat{\Delta}$ of $\widehat{X}$ such that $\widehat{\Delta}_s > c(t_\varepsilon)$ for $0 \leq s < t_\varepsilon$ in the stochastic integral (5.9), we obtain the inequality

$$X_{t_\varepsilon -} \geq \widehat{Y}_{t_\varepsilon -} - \alpha(1+\kappa)c(t_\varepsilon) + \widetilde{X}_{t_\varepsilon -} \tag{5.12}$$

where $\widehat{Y}_{t_\varepsilon -}$ is given by the (compensated) Poissonian integral

$$\widehat{Y}_{t_\varepsilon -} := \int_{[0,t_\varepsilon)\times[0,c(t_\varepsilon)]} xN(\mathrm{d}s, \mathrm{d}x)\,.$$

By a second moment calculation, there is the inequality

$$\begin{aligned} P\left(|\widehat{Y}_{t_\varepsilon-}| > (1/2-\alpha(1+\kappa))c(t_\varepsilon)\right) &\le \frac{E|\widehat{Y}_{t_\varepsilon-}|^2}{(1/2-\alpha(1+\kappa))^2 c^2(t_\varepsilon)} \\ &\le \frac{t_\varepsilon \int_{\{0<x\le c(t_\varepsilon)\}} x^2\Pi(\mathrm{d}x)}{(1/2-\alpha(1+\kappa))^2 c^2(t_\varepsilon)} \\ &\le \frac{t_\varepsilon U_+(c(t_\varepsilon))}{(1/2-\alpha(1+\kappa))^2 c^2(t_\varepsilon)} \\ &\le \frac{\alpha}{(1/2-\alpha(1+\kappa))^2}, \end{aligned}$$

where the last inequality derives from (5.2). By choice of α, the final expression does not exceed 1/2. We conclude that

$$P\left(\widehat{Y}_{t_\varepsilon-} - \alpha(1+\kappa)c(t_\varepsilon) \ge -c(t_\varepsilon)/2\right) \ge 1/2. \tag{5.13}$$

We will also use the fact that $\widetilde{X}$ is a mean zero Lévy process which is spectrally negative (i.e., with no positive jumps), so

$$\liminf_{t\downarrow 0} P(\widetilde{X}_t > 0) \ge 1/16\,;$$

see [9], p. 320. As furthermore $\widetilde{X}$ is independent of $\widehat{Y}_{t_\varepsilon-}$, we conclude from (5.12) and (5.13) that (5.11) holds provided that ε has been chosen small enough. ∎

Now suppose (5.1) fails. The remaining results in Theorem 3.1 require the case $\kappa > 1/2$ of:

Proposition 5.2 *Assume that Y is a spectrally negative Lévy process, has zero mean, and is not of bounded variation. Define, for $y>0$, $\lambda>0$,*

$$W_Y(y) := \int_0^y \int_x^1 z\Pi_Y^{(-)}(\mathrm{d}z)\mathrm{d}x,$$

and

$$J_Y(\lambda) := \int_0^1 \exp\left\{-\lambda\left(\frac{y^{\frac{2\kappa-1}{\kappa}}}{W_Y(y)}\right)^{\frac{\kappa}{1-\kappa}}\right\}\frac{\mathrm{d}y}{y}, \tag{5.14}$$

where $\Pi_Y^{(-)}$ *is the canonical measure of* $-Y$*, assumed carried on* $(0,1]$*, and let* $\lambda_Y^* = \inf\{\lambda > 0 : J_Y(\lambda) < \infty\}$*. Then with probability one, for* $1/2 \le \kappa < 1$*,*

$$\limsup_{t\downarrow 0} \frac{Y_t}{t^\kappa} \begin{cases} = \infty \\ \in (0,\infty) \\ = 0 \end{cases} \quad \textit{according as} \quad \lambda_Y^* \begin{cases} = \infty \\ \in (0,\infty) \\ = 0\,. \end{cases}$$

The proof of Proposition 5.2 requires several intermediate steps. Take Y as described, then it has characteristic exponent

$$\Psi_Y(\theta) = \int_{(0,1]} (\mathrm{e}^{-\mathrm{i}\theta x} - 1 + \mathrm{i}\theta x)\Pi_Y^{(-)}(\mathrm{d}x).$$

So we can work with the Laplace exponent

$$\psi_Y(\theta) = \Psi_Y(-\mathrm{i}\theta) = \int_{(0,1]} (\mathrm{e}^{-\theta x} - 1 + \theta x)\Pi_Y^{(-)}(\mathrm{d}x), \tag{5.15}$$

such that $E\mathrm{e}^{\theta Y_t} = \mathrm{e}^{t\psi_Y(\theta)}$, $t \ge 0$, $\theta \ge 0$.

Let $T = (T_t, t \ge 0)$ denote the first passage process of Y; this is a subordinator whose Laplace exponent Φ is the inverse function to ψ_Y ([1], p.189), and since $Y(T_t) \equiv t$ we see that the alternatives in Proposition 5.2 can be deduced immediately from

$$\limsup_{t\downarrow 0} \frac{Y_t}{t^\kappa} \begin{cases} = \infty \\ \in (0,\infty) \\ = 0 \end{cases} \iff \liminf_{t\downarrow 0} \frac{T_t}{t^{1/\kappa}} \begin{cases} = 0 \\ \in (0,\infty) \\ = \infty. \end{cases}$$

The subordinator T must have zero drift since if $\lim_{t\downarrow 0} T_t/t := c > 0$ a.s. then $\sup_{0<s\le T_t} Y_s = t$ (see [1], p.191) would give $\limsup_{t\downarrow 0} Y_t/t \le 1/c < \infty$ a.s., thus $Y \in bv$, which is not the case. We can assume T has no jumps bigger than 1, and further exclude the trivial case when T is compound Poisson. So the main part of the proof of Proposition 5.2 is the following, which is a kind of analogue of Theorem 1 of Zhang [17].

Lemma 5.2 *Let* T *be any subordinator with zero drift whose Lévy measure* Π_T *is carried by* $(0,1]$ *and has* $\overline{\Pi}_T(0+) = \infty$*, where* $\overline{\Pi}_T(x) = \Pi_T\{(x,\infty)\}$ *for* $x > 0$*. Put* $m_T(x) = \int_0^x \overline{\Pi}_T(y)\mathrm{d}y$ *and for* $d > 0$ *let*

$$K_T(d) := \int_0^1 \frac{\mathrm{d}y}{y} \exp\left\{-d\frac{(m_T(y))^{\frac{\gamma}{\gamma-1}}}{y}\right\}, \text{ where } \gamma > 1. \tag{5.16}$$

Let $d_K^* := \inf\{d > 0 : K_T(d) < \infty\} \in [0, \infty]$. *Then, with probability one,*
(i) $d_K^* = 0$ *iff* $\lim_{t \downarrow 0} \frac{T_t}{t^\gamma} = \infty$*;*
(ii) $d_K^* = \infty$ *iff* $\liminf_{t \downarrow 0} \frac{T_t}{t^\gamma} = 0$*;*
(iii) $d_K^* \in (0, \infty)$ *iff* $\liminf_{t \downarrow 0} \frac{T_t}{t^\gamma} = c$*, for some* $c \in (0, \infty)$*.*

Before beginning the proof of Lemma 5.2, we need some preliminary results. To start with, we need the following lemma.

Lemma 5.3 *Let* $S = (S_t, t \geq 0)$ *be a subordinator, and* a *and* γ *positive constants. Then*

$$\liminf_{t \downarrow 0} \frac{S_t}{t^\gamma} \leq a \text{ a.s.} \tag{5.17}$$

if and only if for every $r \in (0, 1)$ *and* $\eta > 0$

$$\sum_{n \geq 1} P(S_{r^n} \leq (a + \eta) r^{n\gamma} \ ) = \infty.$$

Proof of Lemma 5.3: One way is obvious, so suppose

$$\sum P\left(S_{r^n} \leq a r^{n\gamma}/(1 - r)\right) = \infty. \tag{5.18}$$

For a given $\varepsilon > 0$, define events

$$A_n = \{S_{r^n} - S_{r^{n+1}} \leq a r^{n\gamma}/(1 - r)\}, \ B_n = \{S_{r^{n+1}} \leq \varepsilon r^{(n+1)\gamma}\}, \ n \geq 0.$$

Then the $\{A_n\}_{n \geq 0}$ are independent, and each B_n is independent of the collection $\{A_n, A_{n-1}, \cdots, A_0\}$. Further, $\sum_{n \geq 0} P(A_n) = \infty$ by (5.18) (recall S is a subordinator), so $P(A_n \text{ i.o.}) = 1$. Then, by the Feller–Chung lemma ([4], p. 69) we can deduce that $P(A_n \cap B_n \text{ i.o.}) = 1$, provided $P(B_n)$ is bounded away from 0: $P(B_n) \geq 1/2$, say, for n large enough. To see that this is the case here, take $b > 0$ and $\varepsilon \in (0, 1)$ and truncate the jumps of S (which is of bounded variation) at $b\varepsilon > 0$, where b will be specified more precisely shortly. Thus, let $S_t^\varepsilon = \sum_{0 < s \leq t} \Delta S_s \mathbf{1}_{\{\Delta S_s \leq b\varepsilon)\}}$. Now $S_t - S_t^\varepsilon$ is nonzero only if there is at least one jump in S of magnitude greater than $b\varepsilon$ up till time t, and this has probability bounded above by $t\overline{\Pi}_S(b\varepsilon)$, where Π_S is the Lévy measure

of S. Then by a standard truncation argument, and using a first-moment Markov inequality,

$$\begin{aligned} P(S_t > \varepsilon t^\gamma) &\leq \frac{t\int_{(0,b\varepsilon]} x\Pi_S(\mathrm{d}x)}{\varepsilon t^\gamma} + t\overline{\Pi}_S(b\varepsilon) \\ &\leq \frac{t\int_{(0,b\varepsilon]} x\Pi_S(\mathrm{d}x)}{\varepsilon t^\gamma} + \frac{t\overline{\Pi}_S(b\varepsilon)}{\varepsilon t^\gamma} \quad (\text{once } \varepsilon t^\gamma \leq 1) \\ &= \frac{tm_T(b\varepsilon)}{\varepsilon t^\gamma} \leq \frac{tm_T(b)}{\varepsilon t^\gamma}. \end{aligned}$$

Now choose $b = h(\varepsilon t^{\gamma-1}/2)$, where $h(\cdot)$ is the inverse function to $m_T(\cdot)$. Then the last ratio is smaller than $1/2$. Replacing t by r^{n+1} in this gives $P(B_n) \geq 1/2$ for n large enough. Finally $P(A_n \cap B_n \text{ i.o.}) = 1$ implies $P(S_{r^n} \leq r^{n\gamma}(a/(1-r) + \varepsilon r^\gamma) \text{ i.o.}) = 1$, thus

$$\liminf_{t\downarrow 0} \frac{S_t}{t^\gamma} \leq \frac{a}{1-r} + \varepsilon r^\gamma \text{ a.s.},$$

in which we can let $\varepsilon \downarrow 0$ and $r \downarrow 0$ to get (5.17). ∎

Applying Lemma 5.3 to T_t, we see that the alternatives in (i)–(iii) of Lemma 5.2 hold iff for some $r < 1$, for all, none, or some but not all, $a > 0$,

$$\sum_{n\geq 1} P(T_{r^n} \leq ar^{n\gamma}\ ) < \infty. \tag{5.19}$$

The next step is to get bounds for the probability in (5.19). One way is easy. Since $\overline{\Pi}_T(0+) = \infty$, $\overline{\Pi}_T(x)$ is strictly positive, and thus $m_T(x)$ is strictly increasing, on a neighbourhood of 0. Recall that we write $h(\cdot)$ for the inverse function to $m_T(\cdot)$.

Lemma 5.4 *Let T be a subordinator with canonical measure Π_T satisfying $\overline{\Pi}_T(0+) = \infty$. Then there is an absolute constant K such that, for any $c > 0$ and $\gamma > 0$,*

$$P(T_t \leq ct^\gamma) \leq \exp\left\{-\frac{ct^\gamma}{h(2ct^{\gamma-1}/K)}\right\},\ t > 0. \tag{5.20}$$

Proof of Lemma 5.4: We can write

$$\Phi(\lambda) = -\frac{1}{t}\log E\mathrm{e}^{-\lambda T_t} = \int_{(0,1]} \left(1 - \mathrm{e}^{-\lambda x}\right)\Pi_T(\mathrm{d}x),\ \lambda > 0.$$

Markov's inequality gives, for any $\lambda > 0$, $c > 0$,

$$\begin{aligned} P(T_t \le ct^\gamma) &\le \mathrm{e}^{\lambda ct^\gamma} E(\mathrm{e}^{-\lambda T_t}) \\ &\le \exp\{-\lambda t(\lambda^{-1}\Phi(\lambda) - ct^{\gamma-1})\} \\ &\le \exp\{-\lambda t(Km_T(1/\lambda) - ct^{\gamma-1})\}, \text{ for some } K > 0, \end{aligned}$$

where we have used [1], Prop. 1, p. 74. Now choose $\lambda = 1/h(2ct^{\gamma-1}/K)$ and we have (5.20). ∎

The corresponding lower bound is trickier:

Lemma 5.5 *Suppose that T is as in Lemma 5.2, and additionally satisfies $\lim_{t\downarrow 0} P(T_t \le dt^\gamma) = 0$ for some $d > 0$ and $\gamma > 1$. Then for any $c > 0$*

$$P(T_t \le ct^\gamma) \ge \frac{1}{4}\exp\left\{-\frac{ct^\gamma}{h(ct^{\gamma-1}/4)}\right\} \text{ for all small enough } t > 0. \tag{5.21}$$

Proof of Lemma 5.5: Take $\gamma > 1$ and assume $\lim_{t\downarrow 0} P(T_t \le dt^\gamma) = 0$, where $d > 0$. First we show that

$$\frac{t^\gamma}{h(t^{\gamma-1})} \to \infty \text{ as } t \downarrow 0. \tag{5.22}$$

To do this we write, for each fixed $t > 0$, $T_t = T_t^{(1)} + T_t^{(2)}$, where the distributions of the independent random variables $T_t^{(1)}$ and $T_t^{(2)}$ are specified by

$$\log E\mathrm{e}^{-\lambda T_t^{(1)}} = -t\int_{(0,h(\varepsilon t^{\gamma-1})]} (1 - \mathrm{e}^{-\lambda x})\Pi_T(\mathrm{d}x)$$

and

$$\log E\mathrm{e}^{-\lambda T_t^{(2)}} = -t\int_{(h(\varepsilon t^{\gamma-1}),1]} (1 - \mathrm{e}^{-\lambda x})\Pi_T(\mathrm{d}x),$$

for a given $\varepsilon > 0$. Observe that

$$ET_t^{(1)} = t\int_{(0,h(\varepsilon t^{\gamma-1})]} x\Pi_T(\mathrm{d}x) \le tm_T(h(\varepsilon t^{\gamma-1})) = \varepsilon t^\gamma,$$

so that

$$\begin{aligned} P(T_t > dt^\gamma) &\le P(T_t^{(1)} > dt^\gamma) + P(T_t^{(2)} \ne 0) \\ &\le \frac{ET_t^{(1)}}{dt^\gamma} + 1 - P(T_t^{(2)} = 0) \\ &\le \varepsilon/d + 1 - P(T_t^{(2)} = 0). \end{aligned}$$

Thus for all sufficiently small t,

$$P(T_t^{(2)} = 0) \le \varepsilon/d + P(T_t \le dt^\gamma) \le 2\varepsilon/d,$$

because of our assumption that $\lim_{t\downarrow 0} P(T_t \le dt^\gamma) = 0$. Now

$$P(T_t^{(2)} = 0) = \exp\left(-t\overline{\Pi}_T(h(\varepsilon t^{\gamma-1}))\right)$$

and

$$t\overline{\Pi}_T(h(\varepsilon t^{\gamma-1})) \le \frac{t}{h(\varepsilon t^{\gamma-1})} m_T(h(\varepsilon t^{\gamma-1})) = \frac{\varepsilon t^\gamma}{h(\varepsilon t^{\gamma-1})},$$

so we see, taking say $\varepsilon = d/4$, that $h(t^{\gamma-1}) \le at^\gamma$ for a constant $a > 0$, or, equivalently, $h(t) \le at^{1+\beta}$, where $\beta = 1/(\gamma - 1) > 0$, for all sufficiently small t. However, $m_T(\cdot)$ is concave, so its inverse function h is convex, so $h(t/2) \le h(t)/2 \le a(1/2)^\beta (t/2)^{1+\beta}$, or $h(t) \le a(1/2)^\beta t^{1+\beta}$, for small t. Iterating this argument gives (5.22).

Now write $\eta = \eta(t) = h(ct^{\gamma-1}/4)$ and define processes $(Y_t^{(i)})_{t\ge 0}$, $i = 1, 2, 3$, such that $(T_t)_{t\ge 0}$ and $(Y_t^{(1)})_{t\ge 0}$ are independent, and $(Y_t^{(2)})_{t\ge 0}$ and $(Y_t^{(3)})_{t\ge 0}$ are independent, and are such that $\log E(\mathrm{e}^{-\lambda Y_t^{(i)}}) = -t\int_{(0,1]}(1-\mathrm{e}^{-\lambda x})\Pi_T^{(i)}(\mathrm{d}x)$, $i = 1, 2, 3$, where

$$\begin{aligned}
\Pi_T^{(1)}(\mathrm{d}x) &= \overline{\Pi}_T(\eta)\delta_\eta(\mathrm{d}x), \\
\Pi_T^{(2)}(\mathrm{d}x) &= \overline{\Pi}_T(\eta)\delta_\eta(\mathrm{d}x) + \mathbf{1}_{(0,\eta]}\Pi_T(\mathrm{d}x), \\
\Pi_T^{(3)}(\mathrm{d}x) &= \mathbf{1}_{(\eta,1]}\Pi_T(\mathrm{d}x),
\end{aligned}$$

and $\delta_\eta(\mathrm{d}x)$ is the point mass at η. Then we have $T_t + Y_t^{(1)} \stackrel{d}{=} Y_t^{(2)} + Y_t^{(3)}$, and

$$\begin{aligned}
P(T_t \le ct^\gamma) &\ge P(T_t + Y_t^{(1)} \le ct^\gamma) \\
&\ge P(Y_t^{(3)} = 0)P(Y_t^{(2)} \le ct^\gamma) \\
&= \mathrm{e}^{-t\overline{\Pi}_T(\eta)} P(Y_t^{(2)} \le ct^\gamma).
\end{aligned}$$

Since $t\overline{\Pi}_T(\eta) = \eta^{-1}t\eta\overline{\Pi}_T(\eta) \le \eta^{-1}tm_T(\eta) \le ct^\gamma/h(ct^{\gamma-1}/4)$, (5.21) will follow when we show that $\liminf_{t\downarrow 0} P(Y_t^{(2)} \le ct^\gamma) \ge 1/4$. By construction we have

$$EY_t^{(2)} = t\left(\int_0^\eta x\Pi_T(\mathrm{d}x) + \eta\overline{\Pi}_T(\eta)\right) = tm_T(\eta) = \frac{ct^\gamma}{4},$$

so if we put $Z_t = Y_t^{(2)} - ct^\gamma/4$ and write $t\sigma_t^2 = EZ_t^2$ we can apply Chebychev to get

$$P\left(Y_t^{(2)} \le ct^\gamma\right) \ge P\left(Z_t \le \frac{ct^\gamma}{2}\right) \ge \frac{5}{9}$$

when $t\sigma_t^2 \le c^2t^{2\gamma}/9$. To deal with the opposite case, $t\sigma_t^2 > c^2t^{2\gamma}/9$, we use the Normal approximation in Lemma 4.3. In the notation of that lemma, $m_3 = \int_{|x|\le\eta} |x|^3 \Pi_T^{(2)}(\mathrm{d}x)$ and $m_2 = \sigma_t^2$, in the present situation. Since, then, $m_3 \le \eta\sigma_t^2$, and we have $\eta = h(ct^{\gamma-1}/4) = o(t^\gamma)$, as $t \downarrow 0$, by (5.22), we get

$$\sup_{x\in\mathbb{R}} \left| P(Z_t \le x\sqrt{t}\sigma_t) - 1 + \overline{F}(x) \right| \le \frac{A\eta\sigma_t^2}{\sqrt{t}\sigma_t^3} = o(1), \text{ as } t \downarrow 0.$$

Choosing $x = ct^\gamma/(2\sqrt{t}\sigma_t)$ gives $P(Z_t \le ct^\gamma/2) \ge 1/4$, hence (5.21). ∎

We are now able to establish Lemma 5.2. Again, recall, $h(\cdot)$ is inverse to $m_T(\cdot)$.

Proof of Lemma 5.2: (i) Suppose that $K_T(d) < \infty$ for some $d > 0$ and write $x_n = h(cr^{n(\gamma-1)})$, where $\gamma > 1$ and $0 < r < 1$. Note that since $h(x)/x$ is increasing we have $x_{n+1} \le r^{\gamma-1}x_n$. Also we have $m_T(x_n) = Rm_T(x_{n+1})$ where $R = r^{1-\gamma} > 1$. So for $y \in [x_{n+1}, x_n)$,

$$\begin{aligned} \frac{m_T(y)^{\frac{\gamma}{\gamma-1}}}{y} &\le \frac{m_T(x_n)^{\frac{\gamma}{\gamma-1}}}{x_{n+1}} \\ &= \frac{R^{\frac{\gamma}{\gamma-1}} m_T(x_{n+1})^{\frac{\gamma}{\gamma-1}}}{x_{n+1}} \\ &= \frac{(Rc)^{\frac{\gamma}{\gamma-1}} r^{(n+1)\gamma}}{h(cr^{(n+1)(\gamma-1)})}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{x_{n+1}}^{x_n} \frac{\mathrm{d}y}{y} \exp\left\{-d\frac{m_T(y)^{\frac{\gamma}{\gamma-1}}}{y}\right\} &\ge \exp\left\{-\frac{d(Rc)^{\frac{\gamma}{\gamma-1}} r^{(n+1)\gamma}}{h(cr^{(n+1)(\gamma-1)})}\right\} \log\frac{x_n}{x_{n+1}} \\ &\ge (\log R)\exp\left\{-\frac{c'r^{(n+1)\gamma}}{h(2c'r^{(n+1)(\gamma-1)}/K)}\right\}, \end{aligned}$$

where K is the constant in Lemma 5.4 and we have chosen

$$c = \left(\frac{K}{2d}\right)^{\gamma-1} R^{-\gamma} \text{ and } c' = Kc/2.$$

Then $K_T(d) < \infty$ gives $\sum_1^\infty P(T_{r^n} \le c' r^{n\gamma}) < \infty$, and so $\liminf_{t\downarrow 0} T_t/t^\gamma \ge c' > 0$ a.s., by Lemma 5.3. Thus we see that $\liminf_{t\downarrow 0} T_t/t^\gamma = 0$ a.s. implies that $K_T(d) = \infty$ for every $d > 0$, hence $d_K^* = \infty$.

Conversely, assume that $\liminf_{t\downarrow 0} T_t/t^\gamma > 0$ a.s. Then by Lemma 5.3, $\sum_1^\infty P(T_{r^n} \le c r^{n\gamma}) < \infty$ for some $c > 0$ and $0 < r < 1$. Then $P(T_{r^n} \le c r^{n\gamma}) \to 0$, Lemma 5.5 applies, and we have

$$\sum_1^\infty \exp\left\{-\frac{c r^{n\gamma}}{h(c r^{n(\gamma-1)}/4)}\right\} < \infty.$$

Putting $x_n = h(c r^{n(\gamma-1)}/4)$ (similar to but not the same x_n as in the previous paragraph), and $c' = 4^{\frac{\gamma}{\gamma-1}}/c^{\frac{1}{\gamma-1}}$, we see that

$$\sum_1^\infty \exp\left\{-\frac{c' m_T(x_n)^{\frac{\gamma}{\gamma-1}}}{x_n}\right\} < \infty. \tag{5.23}$$

We have $m_T(x_{n-1}) = R m_T(x_n)$ where $R = r^{1-\gamma} > 1$. Take $L > R$ and let $k_n = \min(k \ge 1 : x_{n-1}L^{-k} \le x_n)$, so that $x_{n-1}L^{-k_n} \le x_n$. Then for any $d > 0$

$$\begin{aligned}
&\int_{x_n}^{x_{n-1}} \exp\left\{-d\frac{m_T(y)^{\frac{\gamma}{\gamma-1}}}{y}\right\} y^{-1}\mathrm{d}y \\
\le\ & \sum_{i=1}^{k_n} \int_{x_{n-1}L^{-i}}^{x_{n-1}L^{1-i}} \exp\left\{-d m_T(y)^{\frac{1}{\gamma-1}} \frac{m_T(y)}{y}\right\} y^{-1}\mathrm{d}y \\
\le\ & \sum_{i=1}^{k_n} \int_{x_{n-1}L^{-i}}^{x_{n-1}L^{1-i}} \exp\left\{-d m_T(x_n)^{\frac{1}{\gamma-1}} \frac{m_T(x_{n-1}L^{-i})}{x_{n-1}L^{-i}}\right\} y^{-1}\mathrm{d}y \\
\le\ & \log L \sum_{i=1}^{\infty} \exp\left\{-d m_T(x_n)^{\frac{1}{\gamma-1}} \frac{L^i m_T(x_n L^{-1})}{x_{n-1}}\right\} \\
\le\ & \log L \sum_{i=1}^{\infty} \exp\left\{-d m_T(x_n)^{\frac{1}{\gamma-1}} \frac{L^{i-1} m_T(x_n)}{x_{n-1}}\right\} \\
=\ & \log L \sum_{i=1}^{\infty} \exp\left\{-d L^{i-1} R^{-\frac{\gamma}{\gamma-1}} \frac{m_T(x_{n-1})^{\frac{\gamma}{\gamma-1}}}{x_{n-1}}\right\}.
\end{aligned}$$

Approximate this last sum with an integral of the form $\int_0^\infty a_n^{Lx}\mathrm{d}x$, where $a_n = \exp(-c' m_T(x_{n-1})^{\frac{\gamma}{\gamma-1}}/x_{n-1})$, with $d = c'LR^{\frac{\gamma}{\gamma-1}} = 4^{\frac{\gamma}{\gamma-1}}Lr^{-\gamma}/c^{\frac{1}{\gamma-1}}$, to see that it is bounded above by a constant multiple of a_n. It follows from (5.23) that $\sum a_n < \infty$, hence we get $K_T(d) < \infty$, and Part (i) follows.

(ii) If $K_T(d) < \infty$ for all $d > 0$ then, because $c' \to 0$ as $d \to \infty$ at the end of the proof of the forward part of Part (i), we have $\liminf_{t\downarrow 0} T_t/t^\gamma = \infty$ a.s., i.e., $\lim_{t\downarrow 0} T_t/t^\gamma = \infty$ a.s. Conversely, if this holds, then because $d \to \infty$ as $c \to 0$ at the end of the proof of the converse part of Part (i), we have $K_T(d) < \infty$ for all $d > 0$. This completes the proof of Lemma 5.2. ∎

Proof of Proposition 5.2: To finish the proof of Proposition 5.2, we need only show that $J_Y(\lambda) = \infty$ for all $\lambda > 0$ is equivalent to $K_T(d) = \infty$ for all $d > 0$, where $K_T(d)$ is evaluated for the first-passage process T of Y, and $\gamma = 1/\kappa$. We have from (5.15), after integrating by parts,

$$\begin{aligned}\psi_Y(\theta) &= \int_0^1 (\mathrm{e}^{-\theta x} - 1 + \theta x)\Pi_Y^{(-)}(\mathrm{d}x)\\ &= \theta\int_0^1 (1-\mathrm{e}^{-\theta x})\overline{\Pi}_Y^{(-)}(x)\mathrm{d}x,\end{aligned}$$

and differentiating (5.15) gives

$$\psi_Y'(\theta) = \int_0^1 x(1-\mathrm{e}^{-\theta x})\Pi_Y^{(-)}(\mathrm{d}x).$$

So we see that $\theta^{-1}\psi_Y(\theta)$ and $\psi_Y'(\theta)$ are Laplace exponents of driftless subordinators, and using the estimate in [1], p.74, twice, we get

$$\psi_Y(\theta) \asymp \theta^2\widetilde{W}_Y(1/\theta) \text{ and } \psi_Y'(\theta) \asymp \theta W_Y(1/\theta),$$

where $\widetilde{W_Y}(x) = \int_0^x A_Y(y)\mathrm{d}y$ and $A_Y(x) := \int_x^1 \overline{\Pi}_Y^{(-)}(y)\mathrm{d}y$, for $x > 0$, we recall the definition of W_Y just prior to (5.14), and " $\asymp$ " means that the ratio of the quantities on each side of the symbol is bounded above and below by finite positive constants for all values of the argument. However, putting $U_Y(x) = \int_0^x 2z\overline{\Pi}_Y^{(-)}(z)\mathrm{d}z$, for $x > 0$, we see that

$$W_Y(x) = \int_0^x\int_y^1 z\Pi_Y^{(-)}(\mathrm{d}z)\mathrm{d}y = \frac{1}{2}U_Y(x) + \widetilde{W}_Y(x),$$

and

$$\widetilde{W}_Y(x) = \frac{1}{2}U_Y(x) + xA_Y(x);$$

thus

$$\widetilde{W}_Y(x) \le W_Y(x) = U_Y(x) + xA_Y(x) \le 2\widetilde{W}_Y(x).$$

Hence we have

$$\theta^2 W_Y(1/\theta) \asymp \psi_Y(\theta) \asymp \theta\psi_Y'(\theta). \tag{5.24}$$

We deduce that $J_Y(\lambda) = \infty$ for all $\lambda > 0$ is equivalent to $\widetilde{J}_Y(\lambda) = \infty$ for all $\lambda > 0$, where

$$\begin{aligned} \widetilde{J}_Y(\lambda) &= \int_0^1 \exp\left\{-\lambda y^{\frac{-1}{1-\kappa}}\psi_Y(1/y)^{\frac{-\kappa}{1-\kappa}}\right\}\frac{\mathrm{d}y}{y} \\ &= \int_1^\infty \exp\left\{-\lambda y^{\frac{1}{1-\kappa}}\psi(y)^{\frac{-\kappa}{1-\kappa}}\right\}\frac{\mathrm{d}y}{y}. \end{aligned}$$

But we know that Φ, the exponent of the first-passage process T, is the inverse of ψ_Y, so making the obvious change of variable gives

$$\widetilde{J}_Y(\lambda) = \int_{\psi_Y(1)}^\infty \exp\left\{-\lambda\Phi(z)^{\frac{1}{1-\kappa}}z^{\frac{-\kappa}{1-\kappa}}\right\}\frac{\Phi'(z)\mathrm{d}z}{\Phi(z)}.$$

From (5.24) we deduce that $z\Phi'(z)/\Phi(z) \asymp 1$ for all $z > 0$, so $J_Y(\lambda) = \infty$ for all $\lambda > 0$ is equivalent to $\widehat{J}_Y(\lambda) = \infty$ for all $\lambda > 0$, where

$$\begin{aligned} \widehat{J}_Y(\lambda) &= \int_1^\infty \exp\left\{-\lambda\Phi(z)^{\frac{1}{1-\kappa}}z^{\frac{-\kappa}{1-\kappa}}\right\}\frac{\mathrm{d}z}{z}, \\ &= \int_0^1 \exp\left\{-\lambda\Phi(z^{-1})^{\frac{1}{1-\kappa}}z^{\frac{\kappa}{1-\kappa}}\right\}\frac{\mathrm{d}z}{z}. \end{aligned}$$

Since $\Phi(z^{-1})$ is bounded above and below by multiples of $z^{-1}m_T(z)$ ([1], p.74), our claim is established. ∎

We are now able to complete the proof of Theorem 3.1:

Proof of Theorem 3.1: The implications (i)⇒(3.1) and (iii)⇒(3.3) stem from Proposition 5.1 and Theorem 2.1, respectively. So we can focus on the situation when

$$\int_0^1 \overline{\Pi}^{(+)}(x^\kappa)\mathrm{d}x < \infty = \int_0^1 \overline{\Pi}^{(-)}(x^\kappa)\mathrm{d}x\,.$$

Recall the decomposition (5.8) where $\widehat{X}_t$ has canonical measure $\Pi^{(+)}(\mathrm{d}x)$. Thus from Theorem 2.1, $\widehat{X}_t$ is $o(t^\kappa)$ a.s. as $t \downarrow 0$. Further, $\widetilde{X}$ is spectrally negative with mean zero. When (ii), (iv) or (v) holds, $X \notin bv$ (see Remark 3 (ii)), so X_t/t takes arbitrarily large positive and negative values a.s. as $t \downarrow 0$, and $\liminf_{t\downarrow 0} X_t/t^\kappa \le 0 \le \limsup_{t\downarrow 0} X_t/t^\kappa$ a.s. The implications (ii)⇒(3.1), (iv)⇒(3.3) and (v)⇒(3.4) now follow from Proposition 5.2. ∎

Remark 5 *Concerning Remark 3 (iii): perusal of the proof of Theorem 2.2 shows that we can add to X a compound Poisson process with masses $f_\pm(t)$, say, at $\pm\sqrt{t}$, provided $\sum_{n\ge 1}\sqrt{t_n} f_\pm(t_n)$ converges, and the proof remains valid. The effect of this is essentially only to change the kind of truncation that is being applied, without changing the value of the limsup, and in the final result this shows up only in an alteration to $V(x)$. Choosing $f(t)$ appropriately, the new $V(\cdot)$ becomes $U(\cdot)$ or $W(\cdot)$, which are thus equivalent in the context of Theorem 2.2. Note that we allow $\kappa = 1/2$ in Proposition 5.2. We will omit further details, but the above shows there is no contradiction with Theorem 2.2.*

5.2 Proof of Theorem 3.2

Theorem 3.2 follows by taking $a(x) = x^\kappa$, $\kappa > 1$, in the Propositions 5.3 and 5.4 below, which are a kind of generalisation of Theorem 9 in Ch. III of [1]. Recall the definition of $A_-(\cdot)$ in (3.5). ∎

Proposition 5.3 *Assume $X \in bv$ and $\delta = 0$. Suppose $a(x)$ is a positive deterministic measurable function on $[0,\infty)$ with $a(x)/x$ nondecreasing and $a(0) = 0$. Let $a^{\leftarrow}(x)$ be its inverse function. Suppose*

$$\int_0^1 \frac{\Pi(\mathrm{d}x)}{1/a^{\leftarrow}(x) + A_-(x)/x} = \infty. \tag{5.25}$$

Then we have

$$\limsup_{t\downarrow 0} \frac{X_t}{a(t)} = \infty \text{ a.s.} \tag{5.26}$$

Proof of Proposition 5.3: Assume X and a as specified. Then the function $a(x)$ is strictly increasing, so $a^{\leftarrow}(x)$ is well defined, positive, continuous, and

nondecreasing on $[0,\infty)$, with $a^{\leftarrow}(0)=0$ and $a^{\leftarrow}(\infty)=\infty$. Note that the function
$$\frac{1}{a^{\leftarrow}(x)}+\frac{A_-(x)}{x}=\frac{1}{a^{\leftarrow}(x)}+\int_0^1\overline{\Pi}^{(-)}(xy)\mathrm{d}y$$
is continuous and nonincreasing, tends to ∞ as $x\to 0$, and to 0 as $x\to\infty$. Choose $\alpha\in(0,1/2)$ arbitrarily small so that

$$2\alpha\left(2(1/2-\alpha)^{-2}+1\right)\le 1,$$

and define, for $t>0$,

$$b(t)=\inf\left\{x>0:\frac{1}{a^{\leftarrow}(x)}+\frac{A_-(x)}{x}\le\frac{\alpha}{t}\right\}.$$

Then $0<b(t)<\infty$ for $t>0$, $b(t)$ is strictly increasing, $\lim_{t\downarrow 0}b(t)=0$, and

$$\frac{t}{a^{\leftarrow}(b(t))}+\frac{tA_-(b(t))}{b(t)}=\alpha. \tag{5.27}$$

Also $b(t)\ge a(t/\alpha)$, and the inverse function $b^{\leftarrow}(x)$ exists and satisfies

$$b^{\leftarrow}(x)=\frac{\alpha}{1/a^{\leftarrow}(x)+A_-(x)/x}. \tag{5.28}$$

Thus, by (5.25),

$$\int_0^1 b^{\leftarrow}(x)\Pi(\mathrm{d}x)=\infty=\int_0^1\overline{\Pi}^{(+)}(b(x))\mathrm{d}x. \tag{5.29}$$

Set
$$U_-(x):=2\int_0^x y\overline{\Pi}^{(-)}(y)\mathrm{d}y.$$
Then we have the upper-bounds

$$t\overline{\Pi}^{(-)}(b(t))\le\frac{tA_-(b(t))}{b(t)}\le\alpha,$$

and
$$\frac{tU_-(b(t))}{b^2(t)}\le\frac{2tA_-(b(t))}{b(t)}\le 2\alpha.$$

Since $X \in bv$ and $\delta = 0$ we can express X in terms of its positive and negative jumps, $\Delta_s^{(+)} = \max(0, \Delta_s)$ and $\Delta_s^{(-)} = \Delta_s^{(+)} - \Delta_s$:

$$X_t = \sum_{0<s\le t} \Delta_s^{(+)} - \sum_{0<s\le t} \Delta_s^{(-)} = X_t^{(+)} - X_t^{(-)}, \text{ say.} \tag{5.30}$$

Recall that $\Delta_s^{(\pm)} \le 1$ a.s. We then have

$$\begin{aligned} & P(X_t^{(-)} > b(t)/2) \\ \le\ & P\left(\sum_{0<s\le t} (\Delta_s^{(-)} \wedge b(t)) > b(t)/2\right) + P\left(\Delta_s^{(-)} > b(t) \text{ for some } s \le t\right) \\ \le\ & P\left(\sum_{0<s\le t} (\Delta_s^{(-)} \wedge b(t)) - tA_-(b(t)) > (1/2-\alpha)b(t)\right) + t\overline{\Pi}^{(-)}(b(t)). \end{aligned}$$

Observe that the random variable $\sum_{0<s\le t}(\Delta_s^{(-)} \wedge b(t)) - tA_-(b(t))$ is centered with variance $tU_-(b(t))$. Hence

$$P\left(\sum_{0<s\le t} (\Delta_s^{(-)} \wedge b(t)) - tA_-(b(t)) > (1/2-\alpha)b(t)\right) \le \frac{tU_-(b(t))}{(1/2-\alpha)^2 b^2(t)},$$

so that, by the choice of α, we finally arrive at

$$P(X_t^{(-)} > b(t)/2) \le \left(\frac{2}{(1/2-\alpha)^2} + 1\right)\alpha \le 1/2. \tag{5.31}$$

By (5.29), $P(X_t^{(+)} > b(t) \text{ i.o.}) \ge P(\Delta_t^{(+)} > b(t) \text{ i.o.}) = 1$. Choose $t_n \downarrow 0$ such that $P(X_{t_n}^{(+)} > b(t_n) \text{ i.o.}) = 1$. Since the subordinators $X^{(+)}$ and $X^{(-)}$ are independent, we have

$$\begin{aligned} & P(X_{t_n} > b(t_n)/2 \text{ i.o.}) \\ & \ge \lim_{m\to\infty} P(X_{t_n}^{(+)} > b(t_n), X_{t_n}^{(-)} \le b(t_n)/2 \text{ for some } n > m) \\ & \ge (1/2) P(X_{t_n}^{(+)} > b(t_n) \text{ i.o.}) \qquad \text{(by (5.31))} \\ & = 1/2. \end{aligned}$$

In the last inequality we used the Feller–Chung lemma ([4], p. 69). Thus $\limsup_{t\downarrow 0} X_t/b(t) \ge 1/2$, a.s. Now since $a(x)/x$ is nondecreasing, we have

for $\alpha < 1$, $b(t)/a(t) \geq a(t/\alpha)/a(t) \geq 1/\alpha$, so $\limsup_{t\downarrow 0} X_t/a(t) \geq 1/\alpha$ a.s. Letting $\alpha \downarrow 0$ gives $\limsup_{t\downarrow 0} X_t/a(t) = \infty$ a.s., as claimed in (5.26). ∎

We now state a strong version of the converse of Proposition 5.3 which completes the proof of Theorem 3.2.

Proposition 5.4 *The notation and assumptions are the same as in Proposition 5.3. If*

$$\int_0^1 \frac{\Pi(\mathrm{d}x)}{1/a^{\leftarrow}(x) + A_-(x)/x} < \infty, \tag{5.32}$$

then we have

$$\limsup_{t\downarrow 0} \frac{X_t}{a(t)} \leq 0 \text{ a.s.} \tag{5.33}$$

We will establish Proposition 5.4 using a coupling technique similar to that in [2]. For this purpose, we first need a technical lemma, which is intuitively obvious once the notation has been assimilated. Let Y be a Lévy process and $((t_i, x_i), i \in I)$ a countable family in $(0,\infty) \times (0,\infty)$ such that the t_i's are pairwise distinct. Let $(Y^i, i \in I)$ be a family of i.i.d. copies of Y, and set for each $i \in I$

$$\rho_i := \inf\left\{s \geq 0 : Y^i_s \geq x_i\right\} \wedge a^{\leftarrow}(x_i),$$

where $a(\cdot)$ is as in the statement of Proposition 5.3. More generally, we could as well take for ρ_i any stopping time in the natural filtration of Y^i, depending possibly on the family $((t_i, x_i), i \in I)$.

Now assume that

$$T_t := \sum_{t_i \leq t} \rho_i \; < \; \infty \text{ for all } t \geq 0 \text{ and } \sum_{i \in I} \rho_i = \infty, \qquad \text{a.s.} \tag{5.34}$$

Then $T = (T_t, t \geq 0)$ is a right-continuous non-decreasing process and (5.34) enables us to construct a process Y' by pasting together the paths $(Y^i_s, 0 \leq s \leq \rho_i)$ as follows. If $t = T_u$ for some (unique) $u \geq 0$, then we set

$$Y'_t \; = \sum_{t_i \leq u} Y^i(\rho_i)\,.$$

Otherwise, there exists a unique $u > 0$ such that $T_{u-} \leq t < T_u$, and thus a unique index $j \in I$ for which $T_u - T_{u-} = \rho_j$, and we set

$$Y'_t \; = \sum_{t_i < u} Y^i(\rho_i) + Y^j(t - T_{u-})\,.$$

Lemma 5.6 *Under the assumptions above, Y' is a version of Y; in particular its law does not depend on the family $((t_i, x_i), i \in I)$.*

Proof of Lemma 5.6: The statement follows readily from the strong Markov property in the case when the family $(t_i, i \in I)$ is discrete in $[0, \infty)$. The general case is deduced by approximation. ∎

We will apply Lemma 5.6 in the following framework. Consider a subordinator $X^{(-)}$ with no drift and Lévy measure $\Pi^{(-)}$; $X^{(-)}$ will play the role of the Lévy process Y above. Let also $X^{(+)}$ be an independent subordinator with no drift and Lévy measure $\Pi^{(+)}$. We write $((t_i, x_i), i \in I)$ for the family of the times and sizes of the jumps of $X^{(+)}$. By the Lévy-Itô decomposition, $((t_i, x_i), i \in I)$ is the family of the atoms of a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\mathrm{d}t \otimes \Pi^{(+)}(\mathrm{d}x)$.

Next, mark each jump of $X^{(+)}$, say (t_i, x_i), using an independent copy $X^{(-,i)}$ of $X^{(-)}$. In other words, $\big((t_i, x_i, X^{(-,i)}), i \in I\big)$ is the family of atoms of a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{D}$ with intensity $\mathrm{d}t \otimes \Pi^{(+)}(\mathrm{d}x) \otimes \mathbb{P}^{(-)}$, where $\mathbb{D}$ stands for the space of càdlàg paths on $[0, \infty)$ and $\mathbb{P}^{(-)}$ for the law of $X^{(-)}$. Finally, define for every $i \in I$,

$$\rho_i := \inf\left\{s \geq 0 : X_s^{(-,i)} \geq x_i\right\} \wedge a^{\leftarrow}(x_i)\,.$$

Lemma 5.7 *In the notation above, the family $((t_i, \rho_i), i \in I)$ fulfills (5.34). Further, the process*

$$T_t := \sum_{t_i \leq t} \rho_i\,, \qquad t \geq 0$$

is a subordinator with no drift.

Proof of Lemma 5.7: Plainly,

$$\sum_{i \in I} \delta_{(t_i, \rho_i)}$$

is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\mathrm{d}t \otimes \mu(\mathrm{d}y)$, where

$$\mu(\mathrm{d}y) := \int_{(0,\infty)} \Pi^{(+)}(\mathrm{d}x)\mathbb{P}^{(-)}\left(\tau_x \wedge a^{\leftarrow}(x) \in \mathrm{d}y\right)\,,$$

and τ_x denotes the first-passage time of $X^{(-)}$ in $[x, \infty)$. So it suffices to check that $\int_{(0,\infty)}(1 \wedge y)\mu(\mathrm{d}y) < \infty$.

In this direction, recall (e.g. Proposition III.1 in [1]) that there is some absolute constant c such that

$$\mathbb{E}^{(-)}(\tau_x) \leq \frac{cx}{A_-(x)}, \qquad \forall x > 0.$$

As a consequence, we have

$$\begin{aligned}\int_{(0,\infty)} y\mu(\mathrm{d}y) &= \int_{(0,\infty)} \Pi^{(+)}(\mathrm{d}x)\mathbb{E}^{(-)}\left(\tau_x \wedge a^{\leftarrow}(x)\right) \\ &\leq \int_{(0,\infty)} \Pi^{(+)}(\mathrm{d}x)\left(\mathbb{E}^{(-)}(\tau_x) \wedge a^{\leftarrow}(x)\right) \\ &\leq c\int_{(0,\infty)} \Pi^{(+)}(\mathrm{d}x)\left(\frac{x}{A_-(x)} \wedge a^{\leftarrow}(x)\right).\end{aligned}$$

Recall that we assume that $\Pi^{(+)}$ has support in $[0,1]$. It is readily checked that convergence of the integral in (5.32) is equivalent to

$$\int_{(0,\infty)} \Pi^{(+)}(\mathrm{d}x)\left(\frac{x}{A_-(x)} \wedge a^{\leftarrow}(x)\right) < \infty.$$

Our claim is established. ∎

We can thus construct a process X', as in Lemma 5.6, by pasting together the paths $(X_s^{(-,i)}, 0 \leq s \leq \rho_i)$. This enables us to complete the proof of Proposition 5.4.

Proof of Proposition 5.4: An application of Lemma 5.6 shows that X' is a subordinator which is independent of $X^{(+)}$ and has the same law as $X^{(-)}$. As a consequence, we may suppose that the Lévy process X is given in the form $X = X^{(+)} - X'$.

Set $Y_t := X'_t + a(t)$. For every jump (t_i, x_i) of $X^{(+)}$, we have by construction

$$\begin{aligned}Y(T_{t_i}) - Y(T_{t_i-}) &= X^{(-,i)}(\rho_i) + a(T_{t_i-} + \rho_i) - a(T_{t_i-}) \\ &\geq X^{(-,i)}(\rho_i) + a(\rho_i) \qquad \text{(as } a(x)/x \text{ increases)} \\ &\geq x_i \qquad \text{(by definition of } \rho_i\text{)}.\end{aligned}$$

By summation (recall that $X^{(+)}$ has no drift), we get that $Y(T_t) \geq X_t^{(+)}$ for all $t \geq 0$.

As $T = (T_t, t \geq 0)$ is a subordinator with no drift, we know from the result of Shtatland (1965) that $T_t = o(t)$ as $t \to 0$, a.s., thus with probability one, we have for every $\varepsilon > 0$

$$X_t^{(+)} \leq X'_{\varepsilon t} + a(\varepsilon t),\ \forall t \geq 0 \text{ sufficiently small}.$$

Since $a(x)/x$ increases, we deduce that for t sufficiently small

$$\frac{X_t}{a(t)} \leq \frac{X_t^{(+)} - X'_{\varepsilon t}}{a(\varepsilon t)/\varepsilon} \leq \varepsilon$$

which completes the proof. ∎

5.3 Proof of Theorem 3.3

Suppose (3.8) holds, so that $X_t > 0$ for all $t \leq$ some (random) $t_0 > 0$. Thus X is irregular for $(-\infty, 0)$ and (3.10) follows from [2]. But [2] has that (3.10) implies $\sum_{0<s\leq t} \Delta_s^{(-)} = o\left(\sum_{0<s\leq t} \Delta_s^{(+)}\right)$, a.s., as $t \downarrow 0$, so, for arbitrary $\varepsilon > 0$, $X_t \geq (1-\varepsilon)\sum_{0<s\leq t} \Delta_s^{(+)} := (1-\varepsilon)X_t^{(+)}$, a.s., when $t \leq$ some (random) $t_0(\varepsilon) > 0$. Now $X_t^{(+)}$ is a subordinator with zero drift. Apply Lemma 5.2 with $\gamma = \kappa$ to get (3.9).

Conversely, (3.10) implies $\sum_{0<s\leq t} \Delta_s^{(-)} = o\left(\sum_{0<s\leq t} \Delta_s^{(+)}\right)$ a.s., and (3.9) and Lemma 5.2 imply $\lim_{t\downarrow 0} \sum_{0<s\leq t} \Delta_s^{(+)}/t^\kappa = \infty$ a.s., hence (3.8). ∎

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